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A convex variational level set model for image segmentation

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ABSTRACT

In this paper, we propose a strictly convex energy functional in a level set formulation for the purpose of two-phase image segmentation. We prove that the value of the unique global minimizer for the energy functional is within the interval $[-1, 1]$ for any image, and equals to 1 in the object and -1 in the background for an ideal binary image. A pointwise convergent numerical scheme is presented to solve the gradient descent flow equation. The proposed model is allowed for flexible initialization and can set a reasonable termination criterion on the algorithm. The proposed model has been successfully applied to some synthesized and real images with promising results.

1. Introduction

Image segmentation is the fundamental step for image analysis and computer vision. The goal of image segmentation is to divide the image domain into different regions so that each region depicts some meaningful objects. Numerous methods have been proposed to solve the image segmentation problem during the last three decades.

Many successful methods for image segmentation involve partial differential equation (PDE)-based and variational level set models. The evolution equation for PDE-based models is directly constructed or indirectly derived from a minimization problem, while the evolution equation for a variational level set model is directly derived from the minimization of energy functional defined on level set functions. So far, there are a lot of researches for PDE-based models [1–7]. In this study, we focus on the variational level set models for image segmentation.

There are a large amount of studies on variational level set models for image segmentation [8–15]. These models have got great success in image segmentation, but the evolution of active contours may result in an unexpected state since the corresponding energy functional has a local minimum. Among a wealth of variational level set models we must mention the active contours without edges model of Chan and Vese [8], one of the most widely used models for two-phase image segmentation. As is the case for many variational level set models, its energy functional to be minimized is non-convex and therefore has local minima. This is a serious difficulty because the local minima of energy functional often provide poor segmentation results. So far, there is an extensive literature on global methods for segmentation in the case in which the two piecewise constant values are known, see [16–18, 21] and references therein. For instance, Chan et al. [16] provided a method to compute global minimizers by showing that solutions could be obtained from a convex relaxation when the two piecewise constant values are known. However, when the piecewise constant values are unknown, the situation becomes more difficult because Chan et al.'s method cannot be applied to globally solve the two-phase piecewise constant segmentation problem. Recently, Brown et al. [18] proposed a completely convex method to tackle this problem, which is guaranteed to compute a global minimizer under certain conditions. However, as is pointed out in [18], these conditions do not hold exactly in practice, so one cannot guarantee to find exact global minimizers of the original Chan–Vese problem. On the other hand, Lee and Seo [19] proposed a modified version of the Chan–Vese

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model (called Lee–Seo model in this paper), which uses two shifted Heaviside functions to circumvent the local minimizers of the energy functional and so obtain a stationary global minimum. Recently, based on the Lee–Seo model, Li and Kim [20] proposed a new variational level set model (called the Li–Kim model in this paper) which can be solved by an accurate and unconditionally stable semi-implicit method.

In this paper, we propose a variational level set model and a pointwise convergent numerical scheme for the purpose of two-phase segmentation. We prove that the energy functional is strictly convex in $L^2(\Omega)$ when the two piecewise constant values are known, which guarantees the existence and uniqueness of global minimizer of the energy functional. We further show that the value of the unique global minimizer is within the interval $[-1,1]$ for an image in $L^2(\Omega)$, and equals to 1 in the object and -1 in the background for an ideal binary image. The pointwise convergence of the proposed numerical scheme is proved analytically. The proposed model is allowed for more flexible initialization; the level set function can be initialized to any function $\phi_0 \in L^2(\Omega)\bigcap L^\infty(\Omega)$ with $\|\phi_0\|_\infty \le 1$, e.g., a constant function that $\phi_0 = 1$.

The organization of the remainder of this paper is as follows. In Section 2, we briefly review three related variational level set models for image segmentation. In Section 3, we propose a new variational level set model and discuss the properties of the proposed energy functional. In Section 4, we discuss numerical method, initialization and termination criterion. Section 5 presents experimental results on synthetic and real images using the proposed model. This paper is summarized in Section 6.

2. Related works

2.1. Chan–Vese model

In [8], Chan and Vese proposed a well-known variational level set model for image segmentation. Let Ω be the two dimensional image domain and I be a given image on Ω. Then, the Chan–Vese model can be expressed as the minimization of the following energy functional:

$$\begin{aligned} E_{CV}(\phi) = \lambda_1 \int_\Omega (I(\mathbf{x}) - c_1)^2 H(\phi(\mathbf{x}))\, d\mathbf{x} \\ + \lambda_2 \int_\Omega (I(\mathbf{x}) - c_2)^2 (1 - H(\phi(\mathbf{x})))\, d\mathbf{x} \\ + \mu \int_\Omega |\nabla H(\phi(\mathbf{x}))|, \end{aligned} \tag{1}$$

where μ, λ_1 and λ_2 are positive parameters, ϕ is a level set function, and H is the Heaviside function defined by

$$H(z) = \begin{cases} 0, & z < 0 \\ 1, & z > 0. \end{cases} \tag{2}$$

The two piecewise constants c_1 and c_2 are defined as

$$c_1 = \frac{\int_\Omega I(\mathbf{x}) H(\phi(\mathbf{x}))\, d\mathbf{x}}{\int_\Omega H(\phi(\mathbf{x}))\, d\mathbf{x}} \quad \text{and} \quad c_2 = \frac{\int_\Omega I(\mathbf{x})(1 - H(\phi(\mathbf{x})))\, d\mathbf{x}}{\int_\Omega (1 - H(\phi(\mathbf{x})))\, d\mathbf{x}}, \tag{3}$$

which represent the mean intensity values of I in $\Omega_1 = \{(x,y) \in \Omega |\ \phi(x,y) > 0\}$ and $\Omega_2 = \{(x,y) \in \Omega | \phi(x,y) < 0\}$, respectively.

The Chan–Vese model works well in processing images with a large amount of noise and detecting objects whose boundaries cannot be defined by gradient. However, the energy functional (1) is non-convex even when the two piecewise constant values are known, and thus has local minima that often provide poor results. Consequently, the contour initialization for this model is very important to obtain satisfactory results [16,17,19]. Besides, even after the zero level set of ϕ stops moving, the value of ϕ still keeps moving to $+\infty$ where it is positive, and to $-\infty$ where it is negative; thus it becomes difficult to set a termination criterion on the algorithm [19].

2.2. Lee–Seo model

To make the solution of image segmentation becomes a stationary global minimum, Lee and Seo [19] proposed the following energy functional with two shifted Heaviside functions:

$$\begin{aligned} E_{LS}(\phi) = \lambda_1 \int_\Omega (I(\mathbf{x}) - c_1)^2 \phi(\mathbf{x}) H(\alpha + \phi(\mathbf{x}))\, d\mathbf{x} \\ - \lambda_2 \int_\Omega (I(\mathbf{x}) - c_2)^2 \phi(\mathbf{x}) H(\alpha - \phi(\mathbf{x}))\, d\mathbf{x}, \end{aligned} \tag{4}$$

where α is an arbitrary small positive value. Here, ϕ is multiplied to prevent from computing a local minimum and $H(\pm\phi)$ is shifted by $\mp\alpha$ to confine the range of ϕ.

As is the case for the Chan–Vese model, the constants c_1 and c_2 are still defined by (3).

The Lee–Seo model has a global minimum, which works well on two-phase image segmentation problems. However, its analytical study (such as the proof of theoretical results) is a very laborious task. Besides, it has time step restrictions on the numerical scheme because of using the explicit Euler's scheme [20].

2.3. Li–Kim model

In [20], Li and Kim replace the Heaviside function in the Lee–Seo model with the function:

$$H_c(z) = \frac{1+z}{2} \tag{5}$$

and get the following functional:

$$\begin{aligned} E_{LK}(\phi) = \lambda_1 \int_\Omega (I(\mathbf{x}) - c_1)^2 \phi(\mathbf{x}) H_c(1 + \phi(\mathbf{x}))\, d\mathbf{x} \\ - \lambda_2 \int_\Omega (I(\mathbf{x}) - c_2)^2 \phi(\mathbf{x}) H_c(1 - \phi(\mathbf{x}))\, d\mathbf{x}, \end{aligned} \tag{6}$$

the two piecewise constants c_1 and c_2 are defined as

$$c_1 = \frac{\int_\Omega I(\mathbf{x}) H_c(\phi(\mathbf{x}))\, d\mathbf{x}}{\int_\Omega H_c(\phi(\mathbf{x}))\, d\mathbf{x}} \quad \text{and} \quad c_2 = \frac{\int_\Omega I(\mathbf{x})(1 - H_c(\phi(\mathbf{x})))\, d\mathbf{x}}{\int_\Omega (1 - H_c(\phi(\mathbf{x})))\, d\mathbf{x}}. \tag{7}$$

The Li–Kim model allows unconditionally stable semi-implicit numerical schemes and works well on two-phase image segmentation. Although some of its properties have been shown via a simple yet heuristic example, it may also be a nontrivial task to give a rigorously analytical study on this model. Besides, the Li–Kim model cannot effectively