European Journal of Mechanics / B Fluids 72 (2018) 293–300

Contents lists available at ScienceDirect

European Journal of Mechanics / B Fluids

journal homepage: www.elsevier.com/locate/ejmflu

Expressions of force and moment exerted on a body in a viscous flow via the flux of vorticity generated on its surface

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ARTICLE INFO

Article history:
Received 31 October 2017
Received in revised form 11 May 2018
Accepted 5 June 2018

ABSTRACT

Nonlinear and viscous contributions in the equation of a vorticity evolution in three-dimensional flow are represented in the divergent form. The concept "vorticity transfer tensor" which describes transference of the vorticity in a flow is introduced. The vortex flux for all the flow points and from the body surface are defined (the vortex flux from the body surface in general case is not the same as the boundary vorticity flux used in the literature). Exact expressions of force and moment via the vortex flux from the body surface for non stationary viscous incompressible flow under the no-slip boundary condition are derived directly from Navier–Stokes equations without applying the conservation law in the whole flow region. The expressions contain only surface integrals, and are valid for calculating force and moment for each body of the system in infinite or bounded space. They are most useful when meshless vortex methods are applied for the flow simulation. That is demonstrated in the numerical example of the flow around the sphere. The formulas obtained can also be useful for checking the accuracy of the computations which are performed in natural variables.

1. Introduction

It is known that the aerodynamic or hydrodynamic forces exerted on a body are closely related to the shed and bound vortices. Nikolai Zhukovsky (or Joukowski) and, independently, Martin Wilhelm Kutta first proved connection of lift force with circulation. The circulation theory was treated and developed in the works of Howarth [1], Sears [2,3], for the case of a potential incident flow of an ideal fluid. In [4] an expression of the hydrodynamic force exerted on a body is obtained for a non-stationary vortex flow of incompressible inviscid fluid. In [5] the hydrodynamic force in viscous flow is expressed via the rate of change of the first vortex momentum, called also as hydrodynamic impulse [6]. The expression of the force obtained in [5] contains the volume integral over all regions of the flow with non-zero vorticity. It allows calculating the force exerted on a single body in infinite space or the total force exerting on the system of bodies but not on each body of the system. Other expressions of the force and moment containing the volume integrals over all regions of the flow with non-zero vorticity have been proposed in [7–11]. In [12] an approximate method of many-body force decomposition was proposed for estimating the force acting on each body of the system.

The relation of forces acting on bodies with the dynamics of vorticity on the surfaces of bodies was studied in [13–15]. The boundary vorticity flux $\boldsymbol{\sigma}$ was defined as

$$\boldsymbol{\sigma} \equiv \nu \frac{\partial \boldsymbol{\omega}}{\partial n} = \boldsymbol{\sigma}_a + \boldsymbol{\sigma}_p + \boldsymbol{\sigma}_{vis} + \boldsymbol{\sigma}_f \tag{1}$$

where $\boldsymbol{\sigma}_a = \boldsymbol{n} \times \boldsymbol{a}$, $\boldsymbol{\sigma}_p = \boldsymbol{n} \times \nabla p/\rho$, $\boldsymbol{\sigma}_{vis} = \nu\,(\boldsymbol{n} \times \nabla) \times \boldsymbol{\omega}$, $\boldsymbol{\sigma}_f = -\boldsymbol{n} \times \boldsymbol{F}_{ex}$ are the boundary vorticity flux constituents caused by the surface acceleration $\boldsymbol{a}$, the tangential pressure gradient, a viscous vortical effect, and external forces respectively; $\boldsymbol{n}$ is normal to the surface, directed into the body; $\boldsymbol{\omega}$ is vorticity.

In [15] the expression of the total force is derived from the conservation law of momentum in the region between the body and the control surface that is made to shrink to the body's surface S_b. Thus the expression was reduced to the surface integral. In the case of three dimension flow it has the form

$$\boldsymbol{F} = -\rho \oint_{S_b} \boldsymbol{r} \times \left(\frac{1}{2}\boldsymbol{\sigma}_p + \boldsymbol{\sigma}_{vis}\right) dS, \quad \boldsymbol{r} \in S_b. \tag{2}$$

The expression of forces via characteristics of the vorticity and velocity fields is most important in the case of using the vortex methods. These methods are based on the vorticity evolution equation obtained by applying the curl operator to the Navier–Stokes equation for incompressible flow with a constant kinematic viscosity ν [16]. These equations do not contain pressure

$$\begin{aligned} &\frac{\partial \boldsymbol{\omega}}{\partial t} = -\nabla \times (\boldsymbol{\omega} \times \boldsymbol{u} + \nu \nabla \times \boldsymbol{\omega}), \\ &\boldsymbol{\omega} = \nabla \times \boldsymbol{u}, \quad \nabla \cdot \boldsymbol{u} = 0 \end{aligned} \tag{3}$$

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https://doi.org/10.1016/j.euromechflu.2018.06.002
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$\boldsymbol{u}$ is velocity, ν is kinematic viscosity.

In the case of plane-parallel flows, Eq. (3) can be written in a divergent form [17]

$$\frac{\partial \omega}{\partial t} = -\nabla \cdot (\boldsymbol{U}\omega), \quad \boldsymbol{U} = \boldsymbol{u} + \boldsymbol{u}_d, \tag{4}$$
$$\boldsymbol{u}_d = -\nu \frac{\nabla \omega}{\omega}.$$

Here vorticity ω is scalar value ($\boldsymbol{\omega} = \omega \boldsymbol{e}_z, \boldsymbol{e}_z$ — unit vector normal to the flow plane), the vector $\boldsymbol{u}_d$, is called as diffusion velocity [17].

The vectors $\boldsymbol{J} = \boldsymbol{u}\omega$ and $\boldsymbol{J}_d = \boldsymbol{u}_d\omega$ describe the advective and diffusion transfer of the vorticity.

In [18,19], the expressions of the pressure force $\boldsymbol{F}_p$, and the moment of forces $\boldsymbol{M}_p$ acting on the profile under the no-slip condition have been obtained

$$\frac{\boldsymbol{F}_p}{\rho} = \boldsymbol{e}_z \times \oint \boldsymbol{r} \, (\boldsymbol{J}_d \cdot \boldsymbol{n}) \, \mathrm{d}l + 2\dot{\boldsymbol{\Omega}} \times \boldsymbol{r}_m S + \ddot{\boldsymbol{r}}_m S$$
$$\frac{\boldsymbol{M}_p}{\rho} = \frac{\boldsymbol{e}_z}{2} \oint \boldsymbol{r}^2 \, (\boldsymbol{J}_d \cdot \boldsymbol{n}) \, \mathrm{d}l + 2\dot{\boldsymbol{\Omega}} I_m - \ddot{\boldsymbol{r}}_m \times \boldsymbol{r}_m S. \tag{5}$$

Here $\boldsymbol{r}_m = \frac{1}{S}\int_{S_b} \boldsymbol{r} ds$ and $I_m = \int_{S_b} r^2 ds$ are the center of gravity and moment of inertia of the region S_b inside the profile respectively; S , $\boldsymbol{\Omega}$ are its area and angular velocity respectively.

Lagrangian vortex methods [17,20] have been developed based on Eq. (4). In these methods the vortex regions of the flow are presented as the set of vortex particles which move at the velocity $\boldsymbol{U}$ with a constant value of circulation. The fluid velocity is determined from the distribution of the vortex particles using the Biot–Savart formula. At each time step, new particles with a circulation Γ_i^{new} are generated at the nodes of the contours of streamlined surfaces, where i is the node number. The values Γ_i^{new} are determined from a system of linear equations expressing the boundary conditions on the surface. The value Γ_i^{new} is connected with the diffusion vortex flux as $\Gamma_i^{new} = \boldsymbol{J}_d \cdot \boldsymbol{n}\Delta t \Delta l$, where $\boldsymbol{n}$ is the unit normal to the contour directed into the flow, and Δl is the length of the contour segment. Thus, the value $\boldsymbol{J}_d \cdot \boldsymbol{n} = \Gamma_i^{new}/(\Delta t \Delta l)$ is known at each time step, and the force and moment can be calculated with help of formulas (5). These formulas allowed us to develop an effective method for solving the fluid–structure interaction problem [19,21,22].

In this work, expressions similar to formulas (5) are derived for three dimensional flows. They are obtained directly from the Navier–Stokes equations without use of the conservation laws and the control surfaces, based on the surface integrals properties only, that make them independent on processes around and the conditions at the external boundaries of the flow.

Application of the formulas is demonstrated on the numerical example of the flow around the sphere. The results are compared with the known experimental and numerical data.

2. Force and moment acting on a body in a three-dimensional incompressible viscous flow

2.1. Pressure force

In this section, derivation of the formula of the pressure force $\boldsymbol{F}_p$ is presented. We use the generalized Stokes theorem for any continuous vector function $\boldsymbol{f}$ with continuous partial derivatives on a curved surface s bounded by a closed contour C

$$\int_s (\boldsymbol{n} \times \nabla) \times \boldsymbol{f} \mathrm{d}s = \oint_C \mathrm{d}\boldsymbol{r} \times \boldsymbol{f} \mathrm{d}l.$$

For the closed surface S this equality transforms to

$$\oint_S (\boldsymbol{n} \times \nabla) \times \boldsymbol{f} \mathrm{d}s = 0 \tag{6}$$

as the closed surface can be presented as the sum of two curved surfaces with common boundary, the contour integrals after summation cancel each other because the integration directions are opposite.

We put $\boldsymbol{f} = \boldsymbol{r}p$ and convert the integrand (6). It can be shown (see Appendix A) that

$$(\boldsymbol{n} \times \nabla) \times (\boldsymbol{r}p) = -2\boldsymbol{n}p - \boldsymbol{r} \times (\boldsymbol{n} \times \nabla p). \tag{7}$$

Substituting (7) in (6) we obtain

$$-2\oint_S p\boldsymbol{n}\mathrm{d}s - \oint_S \boldsymbol{r} \times (\boldsymbol{n} \times \nabla p) \, \mathrm{d}s = 0,$$

i.e.

$$\boldsymbol{F}_p \equiv -\oint_{S_b} p\boldsymbol{n}\mathrm{d}s = \frac{1}{2}\oint_{S_b} \boldsymbol{r} \times (\boldsymbol{n} \times \nabla p) \, \mathrm{d}s. \tag{8}$$

This equality is universal and independent on type of the fluid. It holds for all continuous on the closed surface functions p with piecewise continuous derivatives.

Expressing ∇p from Navier–Stokes equation with the no-slip boundary conditions we obtain at the surface

$$\nabla p = -\rho \frac{D\boldsymbol{u}}{Dt} - \nu\rho\nabla \times \boldsymbol{\omega} + \boldsymbol{F}_{ex} = -\rho\boldsymbol{a} - \nu\rho\nabla \times \boldsymbol{\omega} + \boldsymbol{F}_{ex} \tag{9}$$

where $\boldsymbol{a}$ is acceleration of the surface points, ρ is fluid density, $\boldsymbol{F}_{ex}$ is a mass force acting on the fluid, for example, gravity force $\boldsymbol{F}_{ex} = \rho\boldsymbol{g}$.

Substituting (9) in (8) we obtain.

$$\frac{\boldsymbol{F}_p}{\rho} = -\frac{\nu}{2}\oint_{S_b} \boldsymbol{r} \times (\boldsymbol{n} \times (\nabla \times \boldsymbol{\omega})) \, \mathrm{d}s - \frac{1}{2}\oint_{S_b} \boldsymbol{r} \times (\boldsymbol{n} \times \boldsymbol{a}) \, \mathrm{d}s$$
$$+ \frac{1}{2\rho}\oint_{S_b} \boldsymbol{r} \times (\boldsymbol{n} \times \boldsymbol{F}_{ex}) \, \mathrm{d}s. \tag{10}$$

We introduce the notations

$$\boldsymbol{f}_{ex} = \frac{1}{2}\oint_{S_b} \boldsymbol{r} \times (\boldsymbol{n} \times \boldsymbol{F}_{ex}) \, \mathrm{d}s,$$
$$\boldsymbol{\sigma}_{new} = \nu\boldsymbol{n} \times (\nabla \times \boldsymbol{\omega}).$$

For the gravity force $\boldsymbol{F}_{ex} = \rho\boldsymbol{g}$, $\boldsymbol{f}_{ex}$ is equal to the buoyancy force $\boldsymbol{f}_{ex} = -\boldsymbol{g}\rho V_b$ where V_b is volume of the body. The physical meaning of $\boldsymbol{\sigma}_{new}$ is revealed and compared with (1) in Section 3. In this notation, the expression (10) takes the form

$$\frac{\boldsymbol{F}_p}{\rho} = -\frac{1}{2}\oint_{S_b} \boldsymbol{r} \times \boldsymbol{\sigma}_{new}\mathrm{d}s - \frac{1}{2}\oint_{S_b} \boldsymbol{r} \times (\boldsymbol{n} \times \boldsymbol{a}) \, \mathrm{d}s + \frac{\boldsymbol{f}_{ex}}{\rho}. \tag{11}$$

In the case of a rigid body, the velocity $\boldsymbol{u}_b$ and acceleration $\boldsymbol{a}$ of its points can be written as follows

$$\boldsymbol{u}_b = \dot{\boldsymbol{r}}_m + \boldsymbol{\Omega} \times (\boldsymbol{r} - \boldsymbol{r}_m)$$
$$\boldsymbol{a} = \ddot{\boldsymbol{r}}_m + \dot{\boldsymbol{\Omega}} \times (\boldsymbol{r} - \boldsymbol{r}_m) + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times (\boldsymbol{r} - \boldsymbol{r}_m)) \tag{12}$$

where $\boldsymbol{\Omega}$ is the angle velocity, $\boldsymbol{r}_m$ is the center of mass of a homogeneous body of the same shape. One can show (see Appendix B) that in this case

$$\oint_{S_b} (\boldsymbol{n} \times \boldsymbol{a}) \times \boldsymbol{r}\mathrm{d}s = -2\boldsymbol{r}_m \times \dot{\boldsymbol{\Omega}}V_b + 2\ddot{\boldsymbol{r}}_m V_b.$$

Substituting this in Eq. (11) we obtain

$$\frac{\boldsymbol{F}_p}{\rho} = -\frac{1}{2}\oint_{S_b} \boldsymbol{r} \times \boldsymbol{\sigma}_{new}\mathrm{d}s + \dot{\boldsymbol{\Omega}} \times \boldsymbol{r}_m V_b + \ddot{\boldsymbol{r}}_m V_b + \frac{\boldsymbol{f}_{ex}}{\rho}. \tag{13}$$

2.2. Force of friction

It is known that the friction stress $\boldsymbol{\eta}$ acting on the surface element $\Delta\boldsymbol{s} = \boldsymbol{n}\Delta s$ is