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Analytical solutions for annular multipolar vortex equilibria on a sphere

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ABSTRACT

Analytical solutions of the steady Euler equations on a non-rotating sphere representing stationary multipolar vortices with annular patches of uniform vorticity are constructed. The results are derived by applying a stereographic projection of the surface of the sphere onto the complex plane and then using conformal mapping. They generalise known solutions representing simply connected patches on the sphere and annular patches in the plane.

1. Introduction

The study of vortex dynamics on the surface of a sphere can provide valuable models of flows of planetary atmospheres and oceans which take place on scales large enough for the curvature of the planet's surface to have a significant effect on them [1,2]. However, in comparison with vortex dynamics in the plane, much less is known about vortical flows on a spherical surface. Much rarer still, are known exact equilibrium solutions for such flows. The focus of this paper is the construction of a class of such solutions.

We consider flows on the surface of a sphere of an incompressible, inviscid fluid of constant density (commonly referred to in the literature as the *barotropic model*). For such flows, the simplest, and by far the most well-studied, model of vorticity is the point vortex. Numerous equilibria consisting of configurations of point vortices have been found using a variety of techniques (see, for example, [3–7], as well as further references in [1,2]). More physically realistic than the point vortex are models of distributed vorticity. The simplest such model is the uniform vortex patch. However, equilibrium solutions with distributed vorticity on a sphere are much harder to find. Some results have been obtained numerically, for example those of Polvani and Dritschel [4], which describe corotating rings of identical vortex patches in a background of uniform vorticity covering the rest of the sphere. However, to the best of the author's knowledge, the only known exact equilibrium solutions with distributed vorticity on a sphere are those of Verkley [8,9] (on a rotating sphere), Crowdy and Cloke [10], Crowdy [11], and Alobaidi et al. [12].

The solutions of Crowdy and Cloke [10] consist of a stationary array of multiple point vortices embedded in a single simply connected patch of uniform vorticity of non-trivial shape. They can be considered as the analogues on the sphere of the equilibria presented by Crowdy in [13], which comprise a stationary configuration of point vortices set in a simply connected uniform vortex patch in the plane. Another generalisation of the solutions in [13] are those presented by Crowdy in [14]. These are planar solutions consisting of a stationary array of point vortices embedded in a patch of uniform vorticity where this patch is now *doubly* connected. A natural question to ask is whether one can also find analogues on the sphere of the vorticity distributions presented in [14]. The existence of such solutions is not to be immediately expected, since, as pointed out in [10], other well-known planar results (e.g. the classical rotating elliptical patch solution due to Kirchhoff [15]) do not generalise to a spherical surface. However, in this paper, we shall construct precisely these analogues.

The solutions in [13] are constructed in terms of a parameterising conformal map, which is derived from consideration of what is known as the Schwarz function [16] of the patch boundary. This map is built in terms of rational functions. A similar approach is used to construct the solutions in [14], but now, since the patch is doubly connected, automorphic functions [17] are used to build the parameterising map. The extension to the sphere made in [10]

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is performed by first using a stereographic projection of the surface of the sphere onto the complex plane, and then using an adaptation of the ideas in [13]. We shall adopt a similar approach here, applying a stereographic projection and then adapting the ideas in [14].

One can regard the solutions presented in [10,13,14], and those to be presented here, as being formed from the merger of a set of a more elementary vortical equilibrium solution, namely what is referred to in [13] as a *shielded Rankine vortex*. This consists of a single point vortex embedded at the centre of a circular vortex patch of opposite polarity. The shielded Rankine vortex and the non-trivial generalisations of it presented in [10,13,14] and here, are examples of what are generally referred to as *multipolar vortices* (see, for example, [18]). These are an important class of vorticity distributions which has been widely studied in recent years.

We also mention that the solutions constructed in this paper, like those in [10,13,14], possess an axial symmetry, as do many long-lived vortical structures commonly observed in physical flows [19]. One can regard the solutions constructed in this paper as simple models of bands of vorticity commonly observed in planetary atmospheres.

Finally, we point out that whilst, in general, planetary flows are influenced by the planet's rotation, these effects complicate their analysis considerably and, as in [10], we shall ignore them here.

2. Flows on the surface of a sphere

To begin, let us give a brief description of the formulation we shall use to describe flows on the surface of a sphere. Note that this is identical to that used by Crowdy and Cloke [10] and employs a complex variable.

Without loss of generality, we consider a sphere of unit radius, centred on the origin. We use standard spherical polar coordinates (r, θ, ϕ) (i.e. denoting the radial, latitudinal and longitudinal directions respectively). We consider the flow of an inviscid, incompressible fluid of constant density on the surface, Σ, of the sphere. The velocity field, $\underline{u}$, of the flow, in terms of (r, θ, ϕ), is of the form

$$\underline{u} = (0, u_\theta, u_\phi) \tag{2.1}$$

for some u_θ and u_ϕ. Owing to the incompressibility of the flow, we can introduce a scalar streamfunction Ψ such that $\underline{u} = \nabla\Psi \wedge \underline{e}_r$ where $\underline{e}_r$ denotes the unit vector in the r-direction. Then, one can check that

$$u_\theta = \frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\phi}, \qquad u_\phi = -\frac{\partial\Psi}{\partial\theta}. \tag{2.2}$$

The vorticity field of the flow is given by $\nabla \wedge \underline{u}$. One can check that this equals $\omega\underline{e}_r$, where, in terms of Ψ,

$$\omega = -\nabla^2_\Sigma \Psi, \tag{2.3}$$

where ∇^2_Σ denotes the Laplace–Beltrami operator on Σ, given by

$$\nabla^2_\Sigma = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{2.4}$$

We refer to ω as the scalar vorticity field of the flow.

Finally, we point out that it follows from Gauss' Divergence Theorem that there exists the following constraint on ω:

$$\iint_\Sigma \omega \mathrm{d}\sigma = 0, \tag{2.5}$$

where $\mathrm{d}\sigma$ denotes an element of area of Σ.

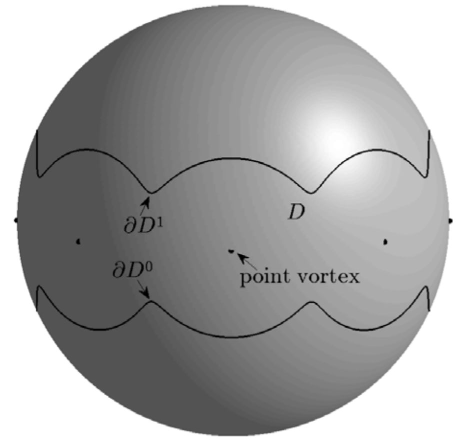

Fig. 1. A doubly connected patch, D, with $K = 8$ point vortices (indicated by dots). ∂D^0 and ∂D^1 are the two components of the boundary of D.

2.1. Complex variable formulation

Now, following Crowdy and Cloke [10], we reformulate the above in terms of a complex variable by using a stereographic projection. This projects Σ onto a complex ζ-plane. The use of a stereographic projection to represent vortex dynamics on a sphere was employed by Dritschel [20]. Much earlier still, it was implemented by Kirchhoff in the related context of flows of electric charge on a spherical surface (see [21] and also [22]). We use a projection mapping the north and south poles of Σ to the point at infinity and the origin, respectively. This projection is given by

$$\zeta = \cot(\theta/2)e^{\mathrm{i}\phi}. \tag{2.6}$$

In terms of this new variable, we write $\Psi(r, \theta, \phi) = \psi(\zeta, \overline{\zeta})$. Then, one can check, as in [10], that in terms of ζ,

$$u_\phi - \mathrm{i}u_\theta = \sqrt{\frac{\zeta}{\overline{\zeta}}}(1 + \zeta\overline{\zeta})\frac{\partial\psi}{\partial\zeta}, \tag{2.7}$$

and

$$\nabla^2_\Sigma \equiv (1 + \zeta\overline{\zeta})\frac{\partial^2}{\partial\zeta\partial\overline{\zeta}}. \tag{2.8}$$

3. A class of vortical equilibria

Following a similar approach to that used by Crowdy and Cloke [10], we shall now construct a class of vortical equilibria on Σ which are the natural analogues on the sphere of the equilibria in [14], namely, consisting of a stationary array of identical point vortices embedded in a patch of uniform vorticity, where this patch is a *doubly* connected domain.

We label our doubly connected patch by D and suppose it to be of uniform vorticity ω_0. We denote the two separate components of the boundary of D by ∂D^j, $j = 0, 1$, and denote the whole boundary of D, consisting of the union of ∂D^0 and ∂D^1, by ∂D. We shall also assume, without loss of generality, that the closure of D does not contain the north pole of the sphere. A schematic example is shown in Fig. 1.

The image in the ζ-plane of D under the stereographic projection (2.6) will be a doubly connected domain, which we shall denote by D_ζ. Also, for $j = 0, 1$, we label the image of ∂D^j by ∂D^j_ζ. Furthermore, we denote by ∂D_ζ the whole boundary of D_ζ, consisting of the union of ∂D^0_ζ and ∂D^1_ζ. We shall also denote the closure of D_ζ by $\overline{D_\zeta}$. It follows from our assumption on D that D_ζ will not