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Two-dimensional thin-film flow of an incompressible inhomogeneous fluid in a channel

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ABSTRACT

We study the 2D flow of an inhomogeneous incompressible fluid in a channel of small aspect ratio. The constitutive equation of the fluid consists of the classical incompressible Navier–Stokes equation plus an extra term depending on the density and its spatial gradient. This type of fluid is particularly suitable for describing the flow of incompressible granular materials. We formulate the mathematical problem and we exploit the smallness of the aspect ratio of the channel to obtain a simplified version (leading order of the lubrication expansion) that can be solved numerically. Choosing a particular set of boundary data we also determine a particular solution that can be compared to the numerical one to prove the consistency of the numerical scheme. The comparison shows good agreement. Finally we extend the model to the case of a channel with variable thickness and we compare our simulations with the ones obtained in Massoudi et al. (2012) [7].

1. Introduction

In 1901 Korteweg [5] presented a model to describe phase transition phenomena in fluids in which the Cauchy stress tensor depends on the density and its spatial gradient. In particular he studied continuous models for fluids that can exist in gaseous and liquid phases, proposing to add an extra contribution to the stress of a Navier–Stokes fluid that depends on the density and on the gradient of the density. The general form of the Cauchy stress tensor $\mathbf{T}$ presented in [5] has the form

$$\mathbf{T} = \Big(-p + \alpha_o|\nabla\varrho|^2 + \alpha_1\Delta\varrho\Big)\mathbf{I} + \alpha_2\Big(\nabla\varrho\otimes\nabla\varrho\Big) + 2\mu\mathbf{D} + \lambda(\mathrm{div}\,\mathbf{v})\mathbf{I} \tag{1.1}$$

where ϱ is the density, $\mathbf{v}$ is the velocity field, $\mathbf{D}$ is the symmetric part of the velocity gradient $\nabla\mathbf{v}$, p is the pressure (which depends on the density) and α_o, α_1, α_2, μ, λ are material moduli that may depend on the density ϱ.

The constitutive equation (1.1) is quite complex if compared to the classical compressible Navier–Stokes system in which $\alpha_j = 0$. In particular one must carefully determine under which conditions the constitutive law (1.1) is consistent with the basic principles of continuum thermodynamics. Many papers have been devoted to this issue (see [1,2,8]), providing more or less sounded thermodynamical frameworks that would be suitable also for possible generalizations. Most of these studies start from a general form of the Cauchy stress $\mathbf{T}$ - like (1.1) or some generalization of it - and then use the second law of thermodynamics to infer restrictions on the material moduli appearing in the definition of $\mathbf{T}$.

A different approach to ensure the consistency of (1.1) with thermodynamical principles is the one adopted in [4]. This approach is based on the ideas of Rajagopal [9] concerning the derivation of constitutive relations from the principle of maximization of entropy production. In this approach, instead of making any assumption on the structure of $\mathbf{T}$, it is prescribed how the system stores energy (Helmholtz free energy) and how entropy is produced. Constitutive equations are then obtained by applying the maximization of entropy production principle. The employment of this procedure has proven to be effective for determining the response of a large class of dissipative materials (see for instance [10–12]) and it has been used in [4] to recover the response (1.1) and some generalizations that include thermal effects and other non-Newtonian phenomena such as shear-thinning and stress relaxation.

Eq. (1.1) was initially introduced in [5] to describe capillarity effects and to model the liquid-vapor transition under static and dynamic conditions. In that paper it was not discussed the possibility that one could use (1.1) to describe the material response of inhomogeneous fluids, i.e. fluids where the density is constant along each particle's path, but not spatially uniform. The model presented in [5] was just an extension of the classical compressible homogeneous Euler fluid to the case that includes higher spatial gradients. Málek and Rajagopal [6] have shown that one can modify (1.1) to model incompressible inhomogeneous fluid-like bodies, such as incompressible granular materials [4].

Granular materials are formed by discrete particles, namely the grains. When the dimensions of the grains are small and the volume fraction of the interstices is filled with a fluid whose density is smaller than that of the grains, the system can be homogenized and treated as a

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continuum. In general, granular materials are modeled as fluid-like continua that are inherently compressible [4]. However, for small grain-size and in special conditions, such as interlocking, they can be safely treated as incompressible. In this case the continuum undergoes isochoric motion (volumes are preserved) but the density is not constant.[1]

To describe the motion of granular materials, in [6] the authors derived the constitutive equations using the maximization of entropy production criterion and assuming that the Helmholtz free energy depends on the density and on its spatial gradient. They named these materials “incompressible inhomogeneous fluids”. They found that the stress **T** depends on the tensor product of the density gradient and other quantities. The model, that is similar to the one proposed by Goodman and Cowin [3] for compressible granular continua, was developed in a general 3D setting and a simple analytical solution of the 1D steady simple shear flow with constant viscosity was determined.

In this paper we extend the model presented in [6] to the case of a bi-dimensional flow in a channel with small aspect ratio. The particular geometry allows one to apply the lubrication scaling and to approximate the general problem with a simplified one in which some terms are neglected because of the smallness of the aspect ratio. To our knowledge this analysis has never been carried out and we believe that it provides a useful tool for studying the characteristics of incompressible inhomogeneous fluids.

Even though the system is studied at the leading order of the lubrication expansion, the mathematical problem is nevertheless complex and can be solved only numerically. The velocity components are written in terms of the density and its first and second derivatives, so that the mathematical problem consists of a very particular nonlinear transport equation for the density. To solve the problem we use a numerical iterative procedure based on a first-order multi-dimensional upwind scheme. A detailed discussion on the boundary conditions for the density is also carried out.

To ensure the validity of the numerical scheme we have considered a special set of boundary conditions where the density at the inlet of the channel and the initial density are small perturbations of the 1D steady solution that has been derived in [6]. In this particular case it is possible to find a “perturbative” solution for the density that can be used for comparison with the numerical solution. The comparison shows good agreement, proving the reliability of the numerical scheme. We remark that we solve the problem not having in mind any particular application. This is why the boundary conditions used to determine the numerical solutions are arbitrarily chosen and not derived from any experimental data. They are selected with the aim of highlighting the density variations with time. A comparison with some computational results available in the literature [7] is also carried out. The model is finally extended to the case of non parallel walls.

The paper develops as follows: in Section 2 we show how to obtain the constitutive equation by means of the maximization of entropy production criterion. Then, in Section 3, we derive the model for the flow in a channel and we discuss the boundary and initial conditions for the unknowns of the problem (Section 4). In Section 5 we reformulate the problem in a non-dimensional form using the lubrication scaling and in Section 6 we consider the leading order approximation. In Section 7 we show the existence of an analytical solution that can be determined under appropriate hypotheses on the data. In Section 8 we present the numerical scheme and the simulations for different types of initial and boundary data. In Section 9 we determine a particular solution when the initial and boundary data for the density are given as small perturbation of the analytical solution. In Section 10 we solve the perturbative problem numerically. In Section 11 we compare the numerical solution with the perturbative one, discussing the agreement between the two. In Sections 12 and 13, we extend the model to the case of non parallel channel walls and finally, in Section 14, we make a comparison with the numerical results of Massoudi et al. [7]. The last section is devoted to conclusions.

2. Thermodynamical setting

In this section we show, following [6], how to derive the constitutive equation for an inhomogeneous incompressible fluid. The results of this section are entirely derived from [6]. The governing equations for an incompressible inhomogeneous body undergoing an isothermal process take the form

$$\begin{cases} \mathrm{div}\,\mathbf{v} = 0, \\ \dot{\varrho} = 0, \\ \varrho\dot{\mathbf{v}} = \mathrm{div}\,\mathbf{T}, \\ \mathbf{T} : \mathbf{D} - \varrho\dot{\psi} = \xi \geqslant 0, \end{cases} \tag{2.1}$$

where the superposed dot stands for differentiation along a particle path (material derivative). In the system (2.1) ψ is the Helmholtz potential and ξ is the rate of dissipation. Eq. $(2.1)_4$ comes from the Clausius–Duhem inequality (second law of thermodynamics). For an incompressible granular material we write [4]

$$\psi = \psi(\varrho, \nabla\varrho), \qquad \xi = \xi(p, \varrho, \mathbf{D} : \mathbf{D}), \tag{2.2}$$

where

$$p = -\frac{1}{3}\,\mathrm{tr}\,\mathbf{T}, \tag{2.3}$$

is the mean normal stress. We require[2]

$$\nabla_{\mathbf{z}}\psi \otimes \nabla\varrho = \nabla\varrho \otimes \nabla_{\mathbf{z}}\psi, \tag{2.4}$$

where $\nabla_{\mathbf{z}}$ means differentiation w.r.t. the components of $\nabla\varrho$

$$\nabla_{\mathbf{z}}\psi = \left(\frac{\partial\psi}{\partial\varrho_x}, \frac{\partial\psi}{\partial\varrho_y}, \frac{\partial\psi}{\partial\varrho_z}\right).$$

Moreover we assume that the rate of dissipation is of the form [6]

$$\xi = 2\mu(p, \varrho, \mathbf{D} : \mathbf{D})\mathbf{D} : \mathbf{D}. \tag{2.5}$$

Applying the gradient operator to $(2.1)_2$ we find

$$\frac{d}{dt}\left(\nabla\varrho\right) = -\nabla\left(\nabla\varrho\cdot\mathbf{v}\right) + \left(\nabla\left(\nabla\varrho\right)\right)\mathbf{v} = -\left(\nabla\mathbf{v}\right)^T\nabla\varrho.$$

As a consequence

$$\dot{\psi} = \frac{d\psi}{dt} = \underbrace{\nabla_{\varrho}\psi\,\dot{\varrho}}_{=0} + \nabla_{\mathbf{z}}\psi\cdot\frac{d}{dt}\left(\nabla\varrho\right) = -\nabla_{\mathbf{z}}\psi\cdot\left[\left(\nabla\mathbf{v}\right)^T\nabla\varrho\right],$$

and, after rearranging the r.h.s., we get

$$\dot{\psi} = -\left(\nabla_{\mathbf{z}}\psi\otimes\nabla\varrho\right) : (\nabla\mathbf{v})^T = -\left(\nabla_{\mathbf{z}}\psi\otimes\nabla\varrho\right) : \mathbf{D},$$

where the last equality is due to the symmetry condition (2.4). Recalling $(2.1)_4$ we find

$$\left[\mathbf{T} + \varrho\left(\nabla_{\mathbf{z}}\psi\otimes\nabla\varrho\right)\right] : \mathbf{D} - \xi = 0. \tag{2.6}$$

Following [6], we look for a tensor **T** that maximizes ξ of the form (2.5) with respect to **D** under the constraints $(2.1)_1$ and (2.6). Using standard methods of constrained maximization we look for extrema of the functional

$$\xi + \lambda_1\left[\left(\mathbf{T} + \varrho\nabla_{\mathbf{z}}\psi\otimes\nabla\varrho\right) : \mathbf{D} - \xi\right] + \lambda_2\,\mathrm{tr}\,\mathbf{D} = 0, \tag{2.7}$$

[1] The material is inhomogeneous in some reference configuration and density is constant along each particle's path.

[2] The assumption (2.4) is a symmetry relation for the Helmholtz free energy [6]. The symmetry is a consequence of the principle of objectivity that provides the dependence of the Helmholtz potential on the gradient of the density only through its modulus.