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Sliding Mode Control of Uncertain Time Delay System using Lambert W Function

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Abstract: This paper presents a robust sliding mode control law for time delay systems with parametric uncertainties and external disturbances. The uncertainties and disturbances are assumed to be matched. The method for designing a switching hyperplane using Lambert W function is proposed for generation of sliding motion in the system. This method includes desired eigenvalue placement for getting a stable sliding manifold. A controller is designed such that system trajectories reach sliding surface in finite time and remain there for all subsequent time. Simulation results are also shown on a numerical example to illustrate the effectiveness of the proposed method.

Keywords: Time-delay Systems, Sliding Mode Control, Linear Systems, Lambert W Function.

1. INTRODUCTION

Time delays frequently appear in numerous industrial systems such as biological systems, chemical processes, economical system and electrical networks. Most of the processes already has time delay in the system's component. In some systems deliberate addition of time delay is also done to improve stability. Presence of time delay term in the system leads to an infinite number of roots of the characteristic equation called transcendental characteristic equation (TCE). Control performance of the time delay system usually degrades due to the presence of these infinite number of roots. So the analysis of time delay system, like designing stabilizing controllers and analyzing stability, using classical methods are difficult. Therefore the analysis and synthesis of time delay systems have been the active research area over the past decades (Kwon et al. (1980)).

Furthermore, controller design for uncertain systems is one of the main topics of research in control theory as most of the control techniques fail to handle uncertainty in the plant effectively. Edwards (1998) has developed robust control methods. In this context, Utkin (1977); Young (1993); Zinober (1994) have done work on sliding mode control, which is a very popular tool for controlling uncertain system because of its property of robustness.

Stability analysis of time delay systems has been a topic of interest from a long time (Luo et al. (1993)). The application of SMC to uncertain time delay systems using different methods have been investigated by many authors. The design approaches were based on delay dependent and delay independent stability. The earlier development on asymptotic stability, dependent on delay had led to many useful results and has been presented in (Chen et al. (1995); Gouaisbout et al. (2002); Gouaisbaut et al. (1999); Xia et al. (2002, 2013)). Delay independent stability using sliding mode control based on the approach of LMI is presented in Xiang et al. (2003). Qu et al. (2013) developed a delay dependent sufficient condition for the design of a stable sliding mode plane in terms of LMI. The controller design methods adopted in above literature used concepts of LMI, Lyapunov Krasovskii function or both.

Lambert W function has been broadly used in many fields. Some of the applications of the Lambert W function are numerical analysis and asymptotic analysis (Corless et al. (1996)). The most important application is in the solution obtained for delay differential equation (DDE). For stability analysis of time delay system, complete solution of delay differential equation is required and Asl et al. (2003) has developed an analytical approach to get the complete solution of DDE based on the concept of the Lambert W function. The solution of DDE obtained using Lambert W function is analogous to the general solution of ODE and the concept of the state transition matrix used in ODEs can be generalized to DDEs using matrix Lambert W function concept (Yi et al. (2010); Ivanoviene et al. (2011); Cepeda-Gomez et al. (2015)).

The main contribution of this paper is the use of Lambert W function to design sliding surface. This method allows us to assign eigenvalue to desired location if possible, which is not possible in LMI based solution.

The paper is organized as follows. Section 2 gives a brief description of the system and some basic assumptions. Section 3 presents the main result of sliding mode control problem, i.e., sliding mode dynamics analysis which includes designing of switching hyperplane using Lambert W function. In Section 4 sliding mode control law is designed. Numerical example to illustrate the effectiveness of the proposed algorithm over LMI is given in Section 5. Finally Section 6 concludes the paper.

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2. SYSTEM DESCRIPTION

Consider the following uncertain time delay system:

$$\dot{x}(t) = (A + \Delta A)x(t) + (A_h + \Delta A_h)x(t-h) + (B + \Delta B)u(t) + w(t) \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}$ is the control input, and $w(t)$ is the external disturbances. Initial conditions are $x(t) = \phi(t)$ for $-h \leq t \leq 0$ and $x(0) = \phi(0) = x_0$. $\phi(t)$ is an absolutely continuous vector function. $h > 0$ is a known constant delay. $A = (a_{ij})$, $A_h = (a_{ij}^h)$ and $B = (b_i)$ are known and constant matrices of appropriate dimensions; $\Delta A = (\delta a_{ij})$, $\Delta A_h = (\delta a_{ij}^h)$ and $\Delta B = (\delta b_i)$ are unknown constant parametrical uncertainties.

The following assumptions are assumed to be valid.

(1) The matched disturbance $w(t)$ should satisfies following conditions

$$w(t) = (B + \Delta B)f(t) \quad (2)$$

and it is bounded by a known constant ρ:

$$\parallel f(t) \parallel \leq \rho \quad (3)$$

(2) The uncertainty matrices ΔA, ΔA_h and ΔB should be matched

$$rank(\Delta A|\Delta A_h|\Delta B|B) = rank(B) \quad (4)$$

and they are lower and upper bounded as

$$\delta a_{ij} \in [p_{ij}^+ \ p_{ij}^-], \delta a_{ij}^h \in [q_{ij}^+ \ q_{ij}^-], \delta b_i \in [r_i^+ \ r_i^-]$$

where p_{ij}^-, p_{ij}^+, q_{ij}^-, q_{ij}^+, r_i^-, r_i^+ $(i, j = 1, 2, ..., n)$ are known constants.

(3) The matrix B has full column rank, and the pair (A_{11}, A_{12}) is controllable.

(4) $c^T(\hat{B} + \Delta\hat{B}) > 0$

3. SLIDING MODE DYNAMICS ANALYSIS

For the purpose of designing sliding surface, the system is transformed into regular form. The transformation used is $\hat{x}(t) = Tx(t)$, where $T \in \mathbb{R}^{n \times n}$ is a nonsingular matrix. So, the transformed system is

$$\dot{\hat{x}}(t) = (\hat{A} + \Delta\hat{A})\hat{x}(t) + (\hat{A}_h + \Delta\hat{A}_h)\hat{x}(t-h) + (\hat{B} + \Delta\hat{B})u(t) + w(t) \quad (5)$$

where $\hat{A} = TAT^{-1}$, $\Delta\hat{A} = T\Delta AT^{-1}$, $\hat{A}_h = TA_hT^{-1}$, $\Delta\hat{A}_h = T\Delta A_hT^{-1}$ and $\hat{B} = TB = \begin{bmatrix} 0_{(n-1)\times 1} \\ B_2 \end{bmatrix}$

Under the assumptions (1) and (2), the above system can be represented as follow

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} + \Delta\hat{A}_{21} & \hat{A}_{22} + \Delta\hat{A}_{22} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} \hat{A}_{h11} & \hat{A}_{h12} \\ \hat{A}_{h21} + \Delta\hat{A}_{h21} & \hat{A}_{h22} + \Delta\hat{A}_{h22} \end{bmatrix} \begin{bmatrix} z_1(t-h) \\ z_2(t-h) \end{bmatrix} + \begin{bmatrix} 0_{(n-1)\times 1} \\ B_2 + \Delta B_2 \end{bmatrix} (u(t) + f(t)) \quad (6)$$

where, $z_1(t) = \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ \vdots \\ \hat{x}_(n-1)(t) \end{bmatrix} \mathbb{R}^{n-1}$, $z_2(t) \in \mathbb{R}$, $\hat{A}_{11} \in \mathbb{R}^{(n-1)\times(n-1)}$, $\hat{A}_{12} \in \mathbb{R}^{(n-1)\times 1}$, $\hat{A}_{21} \in \mathbb{R}^{1\times(n-1)}$ and $\hat{A}_{22} \in \mathbb{R}$.

The sliding surface has an important impact on transient performance and stability analysis of the system. So the analysis and design of the sliding manifold is one of the main issues in sliding mode control. Define a sliding surface function as

$$\sigma(t) = c^T\hat{x}(t) \quad (7)$$

$$\sigma(t) = \begin{bmatrix} k & 1 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

$$\sigma(t) = kz_1(t) + z_2(t)$$

where $c \in \mathbb{R}^n$ is the sliding surface parameter which need to be designed.

The system (6) can be rewritten as

$$\dot{z}_1(t) = \hat{A}_{11}z_1(t) + \hat{A}_{12}z_2(t) + \hat{A}_{h11}z_1(t-h) + \hat{A}_{h12}z_2(t-h) \quad (8)$$

$$\dot{z}_2(t) = (\hat{A}_{21} + \Delta\hat{A}_{21})z_1(t) + (\hat{A}_{22} + \Delta\hat{A}_{22})z_2(t) + (\hat{A}_{h21} + \Delta\hat{A}_{h21})z_1(t-h) + (\hat{A}_{h22} + \Delta\hat{A}_{h22})z_2(t-h) + (B_2 + \Delta B_2)(u(t) + f(t)) \quad (9)$$

When the system trajectories reach onto the sliding surface $\sigma(t) = 0$, i.e.,

$$z_2(t) = -kz_1(t) \quad (10)$$

the sliding mode dynamics is said to be attained.
Substituting (10) in (8) results in reduced order dynamics of the system

$$\dot{z}_1(t) = (\hat{A}_{11} - \hat{A}_{12}k)z_1(t) + (\hat{A}_{h11} - \hat{A}_{h12}k)z_1(t-h) \quad (11)$$

The above equation is a delay differential equation which can be solved using a special function called Lambert W function.

3.1 Lambert W function

Lambert W function is a complex valued inverse function of $f(W) = We^W$. It has an infinite number of branches. Each eigenvalue is associated with a particular branch of the Lambert W function. The rightmost eigenvalue is obtained using principal branch and it determines the stability of the system

Assume a solution $z_1(t) = e^{St}M^I$ for the system (11). Here S is $(n-1)\times(n-1)$ matrix and M^I is constant $(n-1)\times 1$ vector. Substituting this solution in (11) gives

$$Se^{St}M^I = \hat{A}_{11}e^{St}M^I - \hat{A}_{12}ke^{St}M^I + \hat{A}_{h11}e^{S(t-h)}M^I - \hat{A}_{h12}ke^{S(t-h)}M^I$$

$$(S - \hat{A}_{11} + \hat{A}_{12}k - \hat{A}_{h11}e^{-Sh} + \hat{A}_{h12}ke^{-Sh})e^{St}M^I = 0$$

$$S - (\hat{A}_{11} - \hat{A}_{12}k) = (\hat{A}_{h11} - \hat{A}_{h12}k)e^{-Sh} \quad (12)$$

Both side of the above equation is multiplying by

$$he^{hS}e^{-h(\hat{A}_{11} - \hat{A}_{12}k)}$$