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Controllability Analysis of Linear Time-Invariant Descriptor Systems ★

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Abstract: A canonical form for general linear time-invariant descriptor systems has been developed. Using this, it has been proved that complete controllability is equivalent to the reachable controllability plus controllability at infinity for general descriptor systems. Further, it has been proved that complete controllability is invariant under derivative as well as proportional state feedback while strong controllability is preserved under proportional state feedback but is not necessarily retained under derivative feedback. It is noteworthy that the aforesaid results are available for regular descriptor systems. We have extended these results for general descriptor systems. Examples are provided to illustrate the presented theory.

Keywords: Descriptor Systems, Singular Value Decomposition, Controllability, Feedback.

1. INTRODUCTION

Consider a general linear time-invariant continuous descriptor system

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $E, A \in \mathbb{R}^{m\times n}$, and $B \in \mathbb{R}^{m\times r}$. The vectors $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^r$ represent the state vector and the control (input) vector for the system (1). The set of systems of the form (1) is denoted by $\Sigma_{m,n,r}$ and we write $[E\ A\ B] \in \Sigma_{m,n,r}$ to represent a system of the form (1). Systems of the form (1) are called regular descriptor systems if $m = n$ and there exists a $\lambda \in \mathbb{C}$ such that the matrix pencil $(\lambda E - A)$ is invertible, where $\mathbb{C}$ denotes the set of complex numbers otherwise they are called irregular descriptor systems. Further, if $m = n$ and the matrix E is invertible, then the system (1) is called normal system. The theory for normal systems has been well established. Researchers are engaged in extending many results previously known for normal systems to descriptor systems as these systems arise naturally when we model physical problems, see, e.g., Chua et al. (1987); Brenan et al. (1996); Kumar and Daoutidis (1999); Zhang et al. (2012). Many works on descriptor systems assume regularity but we do not assume this condition. The theory presented in this article is devoted to the general class of linear time-invariant continuous descriptor systems of the form (1).

The study of descriptor systems started since the famous work of Rosenbrock (1974) for regular descriptor systems. He introduced an elegant concept namely restricted system equivalent for descriptor systems which has been used till date to study the different properties for descriptor systems. Thereafter, a number of works has been reported for regular descriptor systems. Verghese et al. (1981) have studied the regular descriptor systems in frequency domain. Apart from generalizing the known results for normal systems to descriptor systems, they studied the impulsive nature of the descriptor systems. Yip and Sincovec (1981) studied the solvability and complete controllability for descriptor systems. They presented a beautiful checking criteria for complete controllability of regular descriptor systems. Cobb (1984) studied the controllability properties for regular descriptor systems in distributional set up. More details on regular descriptor systems can be found in Dai (1989a). Further, many works has been reported for various applications in designing of control problems, see, e.g., Bunse-Gerstner et al. (1992); Lovass-Nagy et al. (1994); Boughari and Radhy (2007). These works are also devoted to square descriptor systems. But, in many real world applications, it is not necessary that the number of equations and the variables of interest are the same in number. That is why, there is a growing interest for the analysis and design of general descriptor systems and extending the results of regular descriptor systems to the general case. Geerts (1993) has investigated the different notions of solvability for the general descriptor systems in the space $\mathscr{C}_{imp}$ that is defined as the collection of all possible linear combinations of elements of C^∞ and $\mathscr{C}_{p-imp}$ where C^∞ is the space of all smooth functions and $\mathscr{C}_{p-imp}$ consists of purely impulsive Dirac $\delta-$distributions and its derivatives. Impulse and impulsive mode controllability have been studied in Ishihara and Terra (2001) and Hou (2004), respectively. Zubova (2011) has studied the full controllability for the descriptor systems of the form (1). Campbell et al. (2012) have shown that any linear as well as nonlinear descriptor systems can be regularized in behavioral setting. Berger and Trenn (2014) have presented an improved version of Kalman controllability decomposition for the descriptor systems. Mishra and Tomar (2015) have studied the complete and strong controllability for the system (1). Further, Mishra et al. (2016) have designed a feedback such that the closed loop system is non-singular.

In the current note, the authors are interested in generalizing few results in respect of controllability that are known for regular descriptor systems to general descriptor systems of the form (1). It is well known that complete controllability is equivalent

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to reachable controllability and controllability at infinity for regular descriptor systems. We have generalized this fact for general descriptor systems by coining a canonical form for the system (1) which is solely depend on the singular value decomposition (SVD) of the system matrices implying the numerical stability of the technique. Further, we study the controllability properties under derivative as well as proportional state feedbacks. We show that complete controllability is invariant under derivative as well as proportional state feedbacks while strong controllability is preserved under proportional state feedback but is no more preserved under derivative feedback.

The rest of the paper is organized as follows. The next Section is devoted to the basic definitions and results on controllability for general descriptor systems which will be used in the subsequent Sections. Complete controllability for the system (1) is studied in Section 3. Controllability properties under derivative as well as proportional state feedback are discussed in Section 4. The presented theory has been applied and demonstrated by some examples in Section 5. Section 6 is devoted to the concluding remarks of the paper.

2. PRELIMINARIES

In this section, few basic definitions and results are provided which will be used in deriving the main results of the paper. All the following concepts are taken from a recent survey Berger and Reis (2013).

A trajectory $(x,u):\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^r$ is said to be a solution of (1) if and only if it belongs to the behavior:

$$\mathfrak{B}_{[E\ A\ B]} := \{(x,u)\in W^{1,1}_{loc}(\mathbb{R},\mathbb{R}^n)\times L^1_{loc}(\mathbb{R},\mathbb{R}^r) : (x,u) \text{ satisfies (1) for almost all } t\in\mathbb{R}\},$$

where

$L^1_{loc}(\mathbb{R},\mathbb{R}^r)$:= Locally Lebesgue integrable functions $u:\mathbb{R}\to\mathbb{R}^r$,

and

$$W^{1,1}_{loc}(\mathbb{R},\mathbb{R}^n) := \{x:\mathbb{R}\to\mathbb{R}^n : x,\dot{x}\in L^1_{loc}(\mathbb{R},\mathbb{R}^n)\}.$$

Definition 2.1. The system (1) is called controllable at infinity if and only if

$$\forall\, x_0\in\mathbb{R}^n\ \exists\,(x,u)\in\mathfrak{B}_{[E\ A\ B]} : x(0)=x_0.$$

The system (1) is controllable at infinity if and only if $\mathscr{V}=\mathbb{R}^n$, where $\mathscr{V}$ is the set of all consistent initial conditions for the system (1) and is defined as follows

$$\mathscr{V}=\{x_0\in\mathbb{R}^n : \exists\,(x,u)\in\mathfrak{B}_{[E\ A\ B]} : x(0)=x_0\}.$$

Definition 2.2. The system (1) is called impulse controllable (I-controllable) if and only if

$$\forall\, x_0\in\mathbb{R}^n\ \exists\,(x,u)\in\mathfrak{B}_{[E\ A\ B]} : Ex(0)=Ex_0.$$

The system (1) is I-controllable if and only if $\mathscr{V}_w=\mathbb{R}^n$, where $\mathscr{V}_w$ is the set of all weakly consistent initial conditions for the system (1) and is defined as follows

$$\mathscr{V}_w=\{x_0\in\mathbb{R}^n : \exists\,(x,u)\in\mathfrak{B}_{[E\ A\ B]} : Ex(0)=Ex_0\}.$$

Definition 2.3. The system (1) is called completely controllable (C-controllable) if and only if

$$\exists\, T>0\ \forall\, x_0,x_f\in\mathbb{R}^n\ \exists\,(x,u)\in\mathfrak{B}_{[E\ A\ B]} : x(0)=x_0 \text{ and } x(T)=x_f.$$

Definition 2.4. The system (1) is called strongly controllable (S-controllable) if and only if

$$\exists\, T>0\ \forall\, x_0,x_f\in\mathbb{R}^n\ \exists\,(x,u)\in\mathfrak{B}_{[E\ A\ B]} : Ex(0)=Ex_0 \text{ and } Ex(T)=Ex_f.$$

Definition 2.5. The system (1) is called reachable controllable (R-controllable) if and only if

$$\exists\, T>0\ \forall\, x_0,x_f\in\mathscr{V}\ \exists\,(x,u)\in\mathfrak{B}_{[E\ A\ B]} : x(0)=x_0 \text{ and } x(T)=x_f.$$

Proposition 1. The system (1) is controllable at infinity if and only if

$$\operatorname{rank}[E\ \ B]=\operatorname{rank}[E\ \ A\ \ B]. \tag{2}$$

Proposition 2. The system (1) is C-controllable if and only if condition (2) is satisfied together with

$$\operatorname{rank}[\lambda E-A\ \ B]=\operatorname{rank}[E\ \ A\ \ B],\quad \forall\,\lambda\in\mathbb{C}. \tag{3}$$

Proposition 3. The system (1) is I-controllable if and only if

$$\operatorname{rank}[E\ \ AV_\infty\ \ B]=\operatorname{rank}[E\ \ A\ \ B], \tag{4}$$

where V_∞ spans null space of E.

Proposition 4. The system (1) is S-controllable if and only if both the conditions (3) and (4) are satisfied.

Proposition 5. The system (1) is R-controllable if and only if

$$\operatorname{rank}[\lambda E-A\ \ B]=\operatorname{rank}_{\mathbb{R}(s)}[sE-A\ \ B],\quad \forall\,\lambda\in\mathbb{C}, \tag{5}$$

where RHS represents the maximum rank of the matrix over $\mathbb{R}(s)$: the quotient field of the ring of polynomials with coefficients in $\mathbb{R}$.

Definition 2.6. Two systems $[E_i\ A_i\ B_i]\in\Sigma_{m,n,r},\ i=1,2$, are called restricted system equivalent (or system equivalent) if and only if there exist invertible matrices $W\in\mathbb{R}^{m\times m}$ and $T\in\mathbb{R}^{n\times n}$:

$$[\lambda E_1-A_1\ \ B_1]=W\,[\lambda E_2-A_2\ \ B_2]\begin{bmatrix}T & 0\\ 0 & I_r\end{bmatrix}.$$

It can easily be shown that all the different types of controllability defined above are invariant under restricted system equivalent.

3. COMPLETE CONTROLLABILITY

In this Section, we present our main Theorem on complete controllability. Before that, we prove the following two Lemmas.

Lemma 1. Consider the system $[E\ A\ B]\in\Sigma_{m,n,r}$. Then, there exist orthogonal matrices $M\in\mathbb{R}^{m\times m}$ and $N\in\mathbb{R}^{n\times n}$ such that

$$MEN=\overset{\quad s_0\qquad k}{\begin{bmatrix}E_{11} & E_{12}\\ \Sigma_{E_2} & 0\\ 0 & 0\end{bmatrix}}\begin{matrix}r_0\\ s_0\\ q\end{matrix},\quad MB=\overset{r}{\begin{bmatrix}B_1\\ 0\\ 0\end{bmatrix}}\begin{matrix}r_0\\ s_0\\ q\end{matrix},\ \text{and}$$

$$MAN=\overset{\quad s_0\qquad k}{\begin{bmatrix}A_1^1 & A_2^1\\ A_1^2 & A_2^2\\ A_3^2 & A_4^2\end{bmatrix}}\begin{matrix}r_0\\ s_0\\ q\end{matrix}.$$

Here, the matrix partitions are compatible. The matrix Σ_{E_2} is a diagonal positive definite while matrix B_1 is full row rank. Moreover, $s_0+k=n$ and $r_0+s_0+q=m$.

Proof. Let $\operatorname{rank}B=r_0$. Then, there exists an orthogonal matrix $M_1\in\mathbb{R}^{m\times m}$ such that

$$M_1E=\overset{n}{\begin{bmatrix}E_1\\ E_2\end{bmatrix}}\begin{matrix}r_0\\ s\end{matrix},\quad M_1A=\overset{n}{\begin{bmatrix}A^1\\ A^2\end{bmatrix}}\begin{matrix}r_0\\ s\end{matrix},\ \text{and}\ M_1B=\overset{r}{\begin{bmatrix}B_1\\ 0\end{bmatrix}}\begin{matrix}r_0\\ s\end{matrix},$$