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Polynomial Time Approximation Scheme for the Minimum-weight k-Size Cycle Cover Problem in Euclidean space of an arbitrary fixed dimension

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Abstract: We study the Min-k-SCCP on the partition of a complete weighted digraph by k vertex-disjoint cycles of minimum total weight. This problem is the generalization of the well-known traveling salesman problem (TSP) and the special case of the classical vehicle routing problem (VRP). It is known that the problem Min-k-SCCP is strongly NP-hard and remains intractable even in the geometric statement. For the Euclidean Min-k-SCCP in $\mathbb{R}^d$, we construct a polynomial-time approximation scheme, which generalizes the approach proposed earlier for the planar Min-2-SCCP. For any fixed $c > 1$, the scheme finds a $(1 + 1/c)$-approximate solution in time of $O\big(n^{d+1}(k \log n)^{(O(\sqrt{d}c))^{d-1}} 2^k\big)$.

Keywords: cycle cover of size k, traveling salesman problem (TSP), NP-hard problem, polynomial-time approximation scheme (PTAS).

1. INTRODUCTION

The Cycle Cover Problem (CCP) is a combinatorial optimization problem, which is to find an optimal cover of a given graph by a set of vertex-disjoint cycles. To the best of our knowledge, for the first time, this problem was introduced in the seminal paper by Sahni and Gonzales (Sahni and Gonzales, 1976). Since that time, the CCP and its various modifications were extensively studied in numerous publications (see, e.g. (Bläser and Manthey, 2005; Bläser et al., 2006; Bläser and Siebert, 2001; Chandran and Ram, 2007; Manthey, 2008, 2009; Szwarcfiter and Wilson, 1979)). Since each cycle in a cover can be considered as a tour of some vehicle visiting an appropriate set of clients, the CCP is closely related to the Vehicle Routing Problem (VRP). Moreover, the studied problem is a natural generalization of the well-known Traveling Salesman Problem (TSP), since the Min-1-SCCP is equivalent to the TSP. Below, we propose a brief overview of the related problems and previous work.

The Traveling Salesman Problem (TSP) (Garey and Johnson, 1979) is a classic combinatorial optimization problem, which deals with finding the minimum-cost salesman tour (Hamiltonian cycle) in a given complete weighted graph. There are several special cases of the problem, and some of them, such as the Metric TSP and the Euclidean TSP, are of particular interest. An instance of the Metric TSP is defined by an undirected weighted graph so that edge weights satisfy the triangle inequality. Furthermore, in Euclidean TSP the nodes of the given graph are points in d-dimensional space (for some $d > 1$), and edge weights are Euclidean distances between the incident nodes. It is known (Papadimitriou, 1977) that the TSP is NP-hard even in the Euclidean case; i.e., the optimal solution can not be found in polynomial time unless $P = NP$. Although the TSP is hardly approximable (Sahni and Gonzales, 1976) in the general setting, there are polynomial-time approximation algorithms for some special cases. For instance, the Metric TSP (Christofides, 1975) can be approximated in polynomial time with a ratio of 3/2, and, for the Euclidean TSP, a polynomial-time approximation scheme (Arora, 1998) and a randomized asymptotically correct algorithm (Gimadi, 2008) are developed.

The Vehicle Routing Problem (VRP) (Dantzig and Ramser, 1959) deals with servicing a number of clients (customers) with a fleet of vehicles. In the simplest case (which is also known as the Multiple Traveling Salesmen Problem or the mTSP (Bektas, 2006)), an instance of the VRP is defined by n client locations and one dedicated location (depot). The goal is to find a minimum-cost set of vehicle routes visiting every client only once so that any route starts and finishes at the depot. Surveys of the recent results concerning polynomial-time approximation algorithms and heuristics for several modifications of this problem can be found in (Toth and Vigo, 2001; Golden et al., 2008; Kumar and Panneerselvam, 2012).

The m-Peripatetic Salesmen Problem (m-PSP) (De Kort, 1991; Krarup, 1975) is related to searching for several edge-disjoint salesmen routes optimizing a given objective function (e.g. the total weight of routes, maximum weight, etc.) As for the TSP, so the m-PSP is intractable and

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hardly approximable in the general case, but, for the Euclidean case, polynomial-time asymptotically correct algorithms are known (Gimadi, 2008; Baburin et al., 2009).

Cycle covers of graphs are spanning subgraphs consisting of vertex-disjoint simple cycles and maybe isolated vertices. In (Bläser et al., 2006; Chandran and Ram, 2007), it is shown that cycle covers provide an efficient tool for approximating the TSP, the VRP, and their modifications. Among others, L-cycle covers are mostly discovered (Bläser and Siebert, 2001; Bläser and Manthey, 2005; Bläser et al., 2006; Manthey, 2009; Chandran and Ram, 2007). For a given subset $L \subset \mathbb{N}$, a cycle cover is called L-cycle cover if every containing cycle has the number of edges, which belongs to L. While computing L-covers of the minimum (or maximum) weight is NP-hard (Manthey, 2008), approximation results for multiple restricted cases of the problem are known. For instance, the Min-k-UCC(1, 2) problem where $L = \{k, k+1, \ldots\}$ and edge weights restricted to 1 and 2, can be approximated polynomially within a factor 7/6 for any k (Bläser and Siebert, 2001). Further, in (Manthey, 2009) it is shown that the Metric Min-L-UCC problem, for any fixed L, is polynomial-time approximable within a constant factor (depending on L).

In (Khachay and Neznakhina, 2015), the Minimum-weight k-Size Cycle Cover Problem (Min-k-SCCP) is introduced. This problem is closely related to both the Traveling Salesman, the Vehicle Routing, and the Minimum L-cycle cover problems. In contrast to the Min-L-UCC problem, we restrict not the length of cycles covering the graph but the number of cycles (size of the cover) itself.

Actually, in the Min-k-SCCP, for a fixed natural number k and a given complete weighted digraph (with loops) $G = (V, E, w)$, it is required to find a minimum-weight cover of the set V by k vertex-disjoint cycles. In (Khachay and Neznakhina, 2015), it is shown that the Min-k-SCCP is strongly NP-hard and preserves its intractability even in the geometric statement. For the Metric Min-k-SCCP 2-approximation polynomial-time algorithm is proposed, its approximation ratio and running time do not depend on k. In (Khachai and Neznakhina, 2015a,b), for the Euclidean Min-2-SCCP in $\mathbb{R}^2$, polynomial-time approximation scheme (PTAS) is developed using the approach extending the famous result proposed in (Arora, 1998) for the Euclidean TSP.

In the paper we consider the Euclidean Min-k-SCCP, where the graph G is supposed to be undirected and the weights of its edges are defined by the Euclidean distances between vertices in $\mathbb{R}^d$. Generalizing our earlier result for the Euclidean Min-2-SCCP in $\mathbb{R}^2$ (see, e.g. (Khachay and Neznakhina, 2015)), we propose PTAS for the Min-k-SCCP for any arbitrary fixed dimension $d > 1$ and $k = O(\log n)$.

2. PTAS FOR THE EUCLIDEAN MIN-K-SCCP

It is generally believed that a combinatorial optimization problem has a polynomial-time approximation scheme (PTAS) if, for any fixed $c > 1$, there exists an algorithm, finding a $(1 + 1/c)$-approximate solution of the problem in time bounded by some polynomial of the instance length. Generally speaking, the order and coefficients of this polynomial can be dependent on c.

The general idea of our algorithm generalizes the approach proposed in (Khachai and Neznakhina, 2015a) when developing the PTAS for the Min-2-SCCP on the plane and consists of the following stages

(i). Decomposition of the problem considered into $m \le k$ independent subproblems. Actually, we construct a partition of the given graph G into vertex-disjoint subgraphs $G_1, \ldots, G_m$. Then, for each subgraph G_i, we consider Euclidean Min-q_i-SCCP for some appropriate number q_i such that $\sum_{i=1}^m q_i = k$.

(ii). Reducing each of the subproblems obtained to corresponding *well-rounded* Euclidean Min-q_i-SCCP.

(iii). For each well-rounded instance, constructing a recursive partition of the enclosing hypercube.

(iv). Proof of the claim that, for any $c > 0$, with high probability where exists an $(1 + 1/c)$-approximate k-size cycle cover.

(v). Deterministic construction of $(1 + 1/c)$-approximate k-size cycle cover by means of dynamic programming standard derandomization scheme.

2.1 Decomposition of the problem

Our constructions are based on the well-known geometric Jungs's inequality, establishing the dependence between the diameter D of a bounded set in d-dimensional space and the radius R of a minimal sphere enclosing this set.

$$\frac{1}{2}D \le R \le \left(\frac{d}{2d+2}\right)^{\frac{1}{2}} D.$$

Center of such a sphere is called a *Chebyshev center*.

Using a modification of Kruskal's algorithm (Khachai and Neznakhina, 2015b), we construct a k-minimum spanning forest (k-MSF) consisting of k trees $T_1, T_2, \ldots, T_k$. To each tree T_i, we assign diameter D_i, a Chebyshev center c_i, and the appropriate radius R_i. Then, we clusterize trees into $m \le k$ clusters using single-linkage (Nearest Neighbour) algorithm (see, e.g. (Gan et al., 2007)). Actually, we assign trees T_i and T_j to the same cluster iff

$$\|c_i - c_j\|_2 \le (2k + 1)R$$

for $R = \max\{R_i\colon\ i \in \mathbb{N}_k\}$. Hereinafter, we do not distinguish i-th cluster and its vertex-set, which we denote by S_i. By construction, $S_1 \cup \ldots \cup S_m = V$ and $S_i \cap S_j = \varnothing$ for any $i \ne j$.

Theorem 1. Any minimum-weight k-size cycle cover has no cycles intersecting more than one cluster.

In addition, for any cluster S_i, its diameter D_{S_i} does not exceed $\left(\frac{d}{2d+2}\right)^{1/2}(2k^2 - k + 1)OPT$.

Proof. 1. Assume the contrary. Suppose, some minimum-weight k-size cycle cover contains a cycle including vertices from different clusters S_i and S_j.

Denote this cycle by P. By assumption, P contains at least two edges, which span vertices of different clusters; the total length of these edges is greater than $2(2k-1)R$. The cycle P visits multiple trees among $T_1, \ldots, T_k$. Moreover,