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On a generalized single machine scheduling problem with time-dependent processing times. ⋆

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Abstract: In this paper, a generalized formulation of a classical single machine scheduling problem is considered. A set of n jobs characterized by their release dates, deadlines and a start time-dependent processing time function $p(t)$ has to be processed on a single machine. The objective is to find a Pareto-optimal set of schedules with respect to the criteria $\varphi_{\max}$ and makespan, where $\varphi_{\max}$ is a non-decreasing function depending on the completion times of the jobs. We present an approach that allows to find an optimal schedule with respect to different scheduling criteria, such as the minimization of makespan, lateness or weighted lateness, tardiness and weighted tardiness etc. in time polynomially depending on the number of jobs. The complexity of the presented algorithm is $O(n^3 \max\{\log n, H, P\})$, where H and P are the complexity of calculating $\varphi_j(t)$ and $p(t)$, respectively.

Keywords: scheduling algorithms, single-machine scheduling, polynomial algorithms, makespan, Pareto set.

1. INTRODUCTION

We consider the problem of scheduling a set N of n jobs to be processed on a single machine. Each job is characterized by a *release time* r_j and a *deadline* D_j. The *processing times* of the jobs are defined by a start time-dependent function $p(t)$ such that the term $t + p(t)$ is non-decreasing and the value $p(t)$ can be calculated in P operations for any t. The machine can process only one job at a time. Preemption of processing is not allowed, i.e., the processing of any job started at time t will be completed at time $t + p(t)$. A *schedule* π assigns a *start time* $S_j(\pi)$ to each job j such that $S_j(\pi) \geq r_j$ and $S_j(\pi) \geq S_k(\pi) + p(S_k(\pi))$ for any job k with $S_k(\pi) < S_j(\pi)$. Let us define $C_j(\pi) = S_j(\pi) + p(S_j(\pi))$ as the *completion time* of the job j under the schedule π. We define the set of *feasible schedules* as $\Pi(N)$, where for each $j \in N$ the following inequality holds under the schedule $\pi \in \Pi(N)$:

$$C_j(\pi) < D_j.$$

We denote the *objective function* as $\max \varphi_j$, where for each job $j \in N$ function φ_j is non-decreasing in the completion time C_j and for each value y, it is possible to find the time t' with

$$t' = \min\{t | \varphi_j(t) \geq y\}$$

in H operations. The goal is to find a feasible schedule $\pi \in \Pi(N)$ satisfying

$$\min_{\pi \in \Pi(N)} \max_{j \in N} \varphi_j(\pi). \quad (1)$$

According to the classical 3-field scheduling classification scheme proposed by Graham et al. (1979), this problem can be denoted as $1|r_j, p_j = p(t), D_j|\varphi_{\max}$. Such a problem formulation includes the following special cases:

- $1|r_j, p_j = p(t), D_j|C_{\max}$. $\varphi_j(t) = t$;
- $1|r_j, p_j = p(t)|L_{\max}$. $\varphi_j(t) = t - d_j$;
- $1|r_j, p_j = p(t)|T_{\max}$. $\varphi_j(t) = \max\{0, t - d_j\}$;
- $1|r_j, p_j = p(t), w_j \geq 0|wL_{\max}$. $\varphi_j(t) = w_j(t - d_j)$;

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- $1|r_j, p_j = p(t), w_j \geq 0|wT_{\max}$. $\varphi_j(t) = \max\{0, w_j(t - d_j)\}$,

where w_j is the weight for job j. It should be noted that in all cases the function $p(t)$ can set time intervals of the availability of the machine (time windows).

In this paper, we consider the problem of constructing a Pareto-set of schedules to solve the bi-criteria problem $1|r_j, p_j = p(t), D_j|\varphi_{\max}, C_{\max}$. To the best of our knowledge, there does not exist any solution algorithm for the problem under consideration in the literature. The existing literature about scheduling with time-dependent processing times has been reviewed in Alidaee et al. (1999) and Cheng et al. (2004). The detailed survey of single machine and parallel machine scheduling problems of jobs which have the deterioration and shorting rates presented in S. Gawiejnowicz (2008). A lot of different problems with time-dependent processing times was considered in Yin et al. (2015). The problem of minimizing the makespan for jobs with equal processing times $1|r_j, p_j = p, D_j|C_{\max}$ was considered by Simons (1978) and Garey et al. (1981). A lot of scheduling problems with equal processing times were considered in the survey by Kravchenko and Werner (2011). A polynomial LP algorithm for the problem $P|r_j, p_j = p, D_j|\max\varphi_j(C_j)$, where φ_j is a non-decreasing function for any j, was presented in Kravchenko and Werner (2007). The solution of the bi-criteria problem $1|r_j, p_j = p|L_{\max}, C_{\max}$ was presented in Lazarev et al. (2015).

The remainder of this paper is as follows. In Section 2, an auxiliary problem is formulated and an algorithm for its solution is presented. An algorithm to construct the Pareto set with respect to the criteria $\varphi_{\max}$ and makespan is presented in Section 3. The complexity of the presented algorithms is estimated in Section 4, and in Section 5 we give some concluding remarks.

2. AUXILIARY PROBLEM

Let $O_1(\pi), \ldots, O_n(\pi)$ be the sequence of jobs in which they are processed under the schedule π, and $O(j, \pi)$ be the ordinal number of job $j \in N$ under the schedule π, i.e.,

$$O(j, \pi) = i \quad \Leftrightarrow \quad O_i(\pi) = j.$$

We define a family of sets $F = \{N_0, N_1, \ldots, N_n\}$ such that for each $i = 0, 1, \ldots, n$, N_i is the set of jobs the ordinal number of which must be not larger than i, i.e., we have $N_0 \subseteq N_1 \subseteq \ldots \subseteq N_n$. For any $i = 0, \ldots, n$, we define $\overline{N_i} = N \setminus N_i$. We say that schedule π *satisfies* the family of sets $F = \{N_0, N_1, \ldots, N_n\}$ if for $i = 1, \ldots, n$ and for any job $j \in N_i$, the inequality

$$O(j, \pi) \leq i$$

holds. This implies that for each $i = 1, \ldots, n$, only jobs from the set $\overline{N_{i-1}}$ can be processed under the ordinal number i, i.e.,

$$O_i(\pi) \in \overline{N_{i-1}}. \tag{2}$$

It is obvious that $N_0 = \emptyset$, $N_n = N$ and for each $i = 0, \ldots, n$, the number of jobs, which belong to set N_i is not larger than i, i.e.,

$$|N_i| \leq i. \tag{3}$$

For each job $j \in N$, we define $N(j, F) = i$ if $j \in N_i$ and $j \notin N_{i-1}$, i.e.,

$$N(j, F) = \min_{i=0,\ldots,n} \{i | j \in N_i\}.$$

Let $\Phi(N, F, y) \subseteq \Pi(N)$ be the set of schedules such that $\pi \in \Phi(N, F, y)$ satisfies the given set F and for each job $j \in N$, the inequality

$$\varphi_j(\pi) < y$$

holds. Note that, if $F^0 = \{\emptyset, \ldots, \emptyset, N\}$, then

$$\Phi(N, F^0, +\infty) = \Pi(N).$$

Now let us formulate the auxiliary problem.

Auxiliary problem. *Find a schedule* $\pi(F, y) \in \Phi(N, F, y)$ *satisfying*

$$\min_{\pi \in \Pi(N)} \max_{j \in N} \{C_j(\pi) | \varphi_j(\pi) < y, C_j(\pi) < D_j\}. \tag{4}$$

To solve this problem, we need to prove the following lemma.

Lemma 1. Under the feasible schedule $\pi \in \Pi(N)$, the start time of the first job must satisfy the inequality

$$S_{O_1(\pi)}(\pi) \geq r_{O_1(\pi)} \tag{5}$$

and for each $i = 2, \ldots, n$, the start time of the job $O_i(\pi)$ must satisfy the inequality

$$S_{O_i(\pi)}(\pi) \geq \max\{r_{O_i(\pi)}, S_{O_{i-1}(\pi)}(\pi) + p(S_{O_{i-1}(\pi)}(\pi))\}. \tag{6}$$

Proof. Due to the definition of the schedule π, each job $j \in N$ must satisfy the inequality

$$S_j(\pi) \geq r_j$$

and for any job k with $S_k(\pi) \geq S_j(\pi)$, the inequality

$$S_j(\pi) \geq S_k(\pi) + p(S_k(\pi))$$

must hold. If $j = O_1(\pi)$, then statement (5) is true. If $k = O_{i-1}(\pi)$ and $j = O_i(\pi)$ for $i = 2, \ldots, n$, then inequality (6) is true.

Now let us present an algorithm to solve the auxiliary problem.

AUXILIARY ALGORITHM A:

0. Input data: $N, F, N(1, F), \ldots, N(n, F), y$.
1. For each $j = 1, \ldots, n$ and $i = 0, \ldots, n$, set:
   a) $m := 0$;
   b) $D_j(y) := \min\{\min t | \varphi_j(t) \geq y, D_j\}$;
   c) $N_i^m := N_i$.
2. Assign the ordinal numbers $i = n, \ldots, 1$ to the jobs according to the latest release date rule subject to the inclusion (2):
$$O_i(\pi^m) := \arg\max_{j \in \overline{N_{i-1}^m}} r_j.$$
3. Set the earliest possible start times of the jobs according to Lemma 1:
$$S_{O_1(\pi^m)}(\pi^m) := r_{O_1(\pi^m)};$$
$$S_{O_i(\pi^m)}(\pi^m) := \max\{r_{O_i(\pi^m)}, S_{O_{i-1}(\pi^m)}(\pi^m) + $$
$$+ p(S_{O_{i-1}(\pi^m)}(\pi^m))\}, i = 2, \ldots, n.$$