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Stabilization of Controller-Driven Nonuniformly Sampled Systems via Digital Redesign

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Abstract: This paper deals with the stabilization problem of nonuniformly sampled systems, assuming that the controller/scheduler can select the sampling period. In this setting, we proposed a sampling-period-varying state feedback controller that stabilizes the closed-loop system for arbitrary selection of bounded sampling periods. The proposed controller is based on the digital redesign technique, which also makes the closed-loop sampled system state response close to a pre-selected closed-loop continuous system.

Keywords: Sampled-data control, Nonuniform sampling, Networked control systems, Stability, Digital redesign

1. INTRODUCTION

Study of nonuniformly sampled systems may originate from networked control systems (Suh (2008), Hespanha et al. (2007), Baillieul and Antsaklis (2007)), event based control (Lunze and Lehmann (2010)), or they can be treated as a subset of discrete switching or hybrid systems (Naghshtabrizi et al. (2006)).

The stability results and control techniques are well-established when the sampling time is uniform, i.e. constant for all times (Kuo (1995), Chen and Francis (1995)). But since nonuniformly sampled systems are time-varying, finding stabilizing controllers for these systems is proved to be difficult and many efforts have been made in recent years Lin and Antsaklis (2009), Kao and Fujioka (2013), Fujioka and Nakai (2010), Seuret (2012), Zhang et al. (2001).

While most of the literature focus on stabilizing a nonuniformly sampled system when the sampling intervals are unknown using a time-invariant controller, we consider a different setting. We assume that the controller/scheduler also selects the next sampling interval for the system online. This setting is also discussed in Haimovich and Osella (2013) and stabilizing controllers are found when the number of inputs is at least 1 less than the unstable eigenvalues of the open loop system.

In this paper, we propose a simple sampling-period-varying state feedback controller that guarantees the stability for any selection of sampling periods from a bounded interval, inspired by the idea of digital redesign of continuous-time controllers.

Digital redesign idea has been around for some time (Kuo (1995), Shieh et al. (1991)) and further studied later such as in Grüne et al. (2008). In this technique, the designer first designs a continuous-time controller for the desired characteristics and then a digital redesign calculation has been done such that the discrete closed loop response is close to the continuous closed loop response.

Although there are some recent works using digital redesign for intermittent sampling (Mirkin (2015)), there is no such work considering our setting to the best of our knowledge.

1.1 Notation and Terminology

For $x \in \mathbb{C}^n$, $\|x\|$ denotes any vector norm and for $A \in \mathbb{C}^{n\times n}$, $\|A\|$ denotes a norm induced by some vector norm. For an invertible $A \in \mathbb{C}^{n\times n}$, $\kappa(A) := \|A\| \left\|A^{-1}\right\|$ is the condition number for some induced norm.

2. PROBLEM FORMULATION

Consider the continuous time system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $\operatorname{rank} B = m \le n$, and (A, B) is stabilizable. Suppose a stabilizing state feedback controller $K \in \mathbb{R}^{m\times n}$ is found via some method like eigenvalue assignment, LQR optimization, etc. So the closed loop system becomes

$$\dot{x}(t) = (A + BK)x(t) \tag{2}$$

Define the sequence of time instances $\{t_k\}_{k\in\mathbb{N}}$ where

$$0 = t_0 < t_1 < \cdots < t_k < \ldots$$

with $\lim_{k\to\infty} t_k = \infty$. We assume that $\{t_k\}_{k\in\mathbb{N}}$ are known a priori and sampling intervals $h_k := t_{k+1} - t_k$ are bounded, i.e. $h_k \in [h_{\min}, h_{\max}], \forall k \in \mathbb{N}$ for some $h_{\max} > h_{\min} > 0$.

The objective is to design a sampled state feedback controller $K(t_k) \in \mathbb{R}^{m\times n}$ such that the system

$$\dot{x}(t) = Ax(t) + BK(t_k)x(t_k), \quad \forall t \in [t_k, t_{k+1}) \tag{3}$$

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is stable and has a similar state response as (2) for any selection of $\{t_k\}_{k\in\mathbb{N}}$ as long as $h_k \in [h_{\min}, h_{\max}]$. Another objective is maximizing $h_{\max}$ while guaranteeing the stability of (3).

Consider the discretized model of (1)

$$x_{k+1} = F_k x_k + G_k u_k \tag{4}$$

where $u_k := u(t_k)$,

$$F_k := e^{Ah_k} \quad \text{and} \quad G_k := \left(\int_0^{h_k} e^{A\tau} d\tau\right) B.$$

The following result is standard:

Lemma 1. Define $K_k := K(t_k)$ where $K(t_k)$ is given in (3). Then the solutions of (3) and

$$x_{k+1} = (F_k + G_k K_k) x_k \tag{5}$$

are same at the sampling instances, i.e. $x_k = x(t_k), \forall k \in \mathbb{N}$, assuming $x(t_0) = x_0$.

Proof. We are going to prove with induction. $x(t_0) = x_0$ by assumption. Assume that $x(t_k) = x_k$, so

$$\begin{aligned} x(t_{k+1}) &= e^{A(t_{k+1}-t_k)} x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\eta)} Bu(\eta) d\eta \\ &= e^{Ah_k} x(t_k) + \left(\int_0^{h_k} e^{A\tau} d\tau\right) Bu(t_k) \\ &= F_k x_k + G_k K_k x_k \\ &= x_{k+1} \end{aligned}$$

So the problem becomes finding K_k and $h_{\max}$ such that system (5) is guaranteed to be stable for $h_k \in [h_{\min}, h_{\max}]$ and the state response of (5) is similar to that of (2) at the sampling instances.

3. STABILITY RESULTS

It turns out that stability conditions for nonuniformly sampled systems are not trivial (Liberzon and Morse (1999)). There are many sufficient stability results exist, such as Lin and Antsaklis (2009), Kao and Fujioka (2013), Seuret (2012), Liberzon et al. (1999).

We use the following definition and sufficient condition for stability.

Definition 2. The system

$$x_{k+1} = F_k x_k \tag{6}$$

is exponentially stable if there exists $c \geq 1$ and $0 < \gamma < 1$ such that

$$\|x_k\| \leq c\gamma^k \|x_0\| \quad \forall k \in \mathbb{N}$$

for some norm $\|\cdot\|$.

Theorem 3. If there exists an induced matrix norm $\|\cdot\|$ such that $\|F_k\| < 1, \forall k \in \mathbb{N}$, then the system (6) is exponentially stable.

Proof. Select $1 > \gamma = \sup_{k\in\mathbb{N}} \|F_k\|$ and $c = 1$. So

$$\begin{aligned} \|x_k\| &= \|F_{k-1} \dots F_1 F_0 x_0\| \\ &\leq \|F_{k-1}\| \dots \|F_1\| \|F_0\| \|x_0\| \\ &\leq \gamma^k \|x_0\| \end{aligned}$$

4. BACKGROUND

Before presenting the main results, we give some useful definitions and lemmas in this section.

Definition 4. We define the matrix function

$$\Theta(A) := \sum_{i=0}^{\infty} \frac{A^i}{(i+1)!} = I + \frac{A}{2!} + \frac{A^2}{3!} + \dots \tag{7}$$

Lemma 5. The following equalities are satisfied for all $A \in \mathbb{R}^{n\times n}$ and $h \in \mathbb{R}^{>0}$

(i) $A\Theta(A) = \Theta(A)A$
(ii) $e^A = I + A\Theta(A)$
(iii) $\int_0^h e^{A\tau} d\tau = h\Theta(Ah)$
(iv) $\Theta(T^{-1}AT) = T^{-1}\Theta(A)T$ for some invertible T.

Proof. Proof is trivial.

Lemma 6. For any $A \in \mathbb{R}^{n\times n}$ and $h \in \mathbb{R}^{>0}$, $\Theta(Ah)$ has an inverse if and only if A does not have any eigenvalue of the form $2k\pi i/h$ for all nonzero integers k.

Proof. Since $\Theta(\cdot)$ is analytic, if λ is an eigenvalue of A, then $\Theta(\lambda h)$ is an eigenvalue of $\Theta(Ah)$. Suppose $\lambda = 0$, so $\Theta(\lambda h) = 1$. Now suppose $\lambda \neq 0$, so

$$\Theta(\lambda h) = \frac{e^{\lambda h} - 1}{\lambda h} \tag{8}$$

using the previous lemma. Since $\lambda h \neq 0$

$$\Theta(\lambda h) = 0 \Leftrightarrow e^{\lambda h} = 1 \Leftrightarrow \lambda h = 2k\pi i, \quad \forall k \in \mathbb{Z} - \{0\}.$$

This concludes the proof.

Corollary 7. $\Theta(Ah)$ has an inverse if h is non-pathological, i.e. $\lambda - \gamma \neq 2k\pi i/h$ for all nonzero integers k where λ and γ are a pair of eigenvalues of A.

Proof. Suppose $\Theta(Ah)$ is singular. Then, A has an eigenvalue $\lambda = 2k\pi i/h$ for some $k \in \mathbb{Z} - \{0\}$. But since A is real, $\bar{\lambda} = -2k\pi i/h$ is also an eigenvalue. This means $\lambda - \bar{\lambda} = 4k\pi i/h$, i.e. h is pathological.

Lemma 8. Let $\|\cdot\|$ be an induced matrix norm. Then for any $A \in \mathbb{C}^{n\times n}$

$$\|A\|_T := \left\|T^{-1}AT\right\| \tag{9}$$

is also an induced norm, where $T \in \mathbb{C}^{n\times n}$ is an invertible matrix.

Proof. Define the vector norm

$$\|x\|_T := \left\|T^{-1}x\right\|$$

for all $x \in \mathbb{C}^n$. It is easy to show that this definition satisfies norm axioms. Then,

$$\begin{aligned} \max_{x\in\mathbb{C}^n} \frac{\|Ax\|_T}{\|x\|_T} &= \max_{y\in\mathbb{C}^n} \frac{\|ATy\|_T}{\|Ty\|_T} \\ &= \max_{y\in\mathbb{C}^n} \frac{\left\|T^{-1}ATy\right\|}{\|T^{-1}Ty\|} \\ &= \|A\|_T \end{aligned}$$

5. MAIN RESULTS

We propose the following controller function $K : [h_{\min}, h_{\max}] \to \mathbb{R}^{m\times n}$

$$\begin{aligned} K(h) := {} & [X(h)B]^{-1} X(h) \\ & \left(\int_0^h e^{A\tau} d\tau\right)^{-1} \left(e^{(A+BK)h} - e^{Ah}\right) \end{aligned} \tag{10}$$

where $X(h) \in \mathbb{R}^{m\times n}$ is arbitrary such that $X(h)B$ has an inverse for all $h \in [h_{\min}, h_{\max}]$. Assuming $\operatorname{rank} B = m$ and non-pathological sampling, $K(h)$ always exists.