Available online at www.sciencedirect.com

ScienceDirect

IFAC-PapersOnLine 49-7 (2016) 574–578

Asymptotic Tracking of Periodic Operation Based on Control Contraction Metrics ⋆

Ruigang Wang * **Jie Bao** *

* *School of Chemical Engineering, The University of New South Wales, Sydney, NSW 2052, Australia (e-mail: j.bao@unsw.edu.au).*

Abstract: Periodic operations can be beneficial to chemical and biological systems, leading to better efficiency and plant economy compared to steady state operation. In this paper, the tracking control of desired periodic operation based on control contraction metrics is considered. A relaxation method is proposed to convert the nonconvex control synthesis problem into a convex sum-of-squares programming. Finally, an application to a Lotka-Volterra system of sustained oscillating chemical reactions is presented for illustration. The main advantages of this approach include: (1) the feedback information is geodesics, which is a general deviation considering local nonlinearity; (2) the differential state feedback control law is reference trajectory independent; (3) the control design can be carried out via numerical optimization.

Keywords: Periodic operation, Control contraction metrics, Sums-of-squares, Lotka-Volterra system

1. INTRODUCTION

Periodic control can be beneficial to operations of chemical reactors (Bailey (1974); Silveston (1987); Silveston and Hudgins (2012)), such as hydrogenation processes, catalytic oxidation, combustion systems and electrochemical process. In biological systems, oscillations are observed in all types of organisms from the simplest to the most complex (Rapp (1987)). As shown in those survey papers (Silveston (1987); Rapp (1987)), periodic operation is more efficient and economic than steady state operation. In the context of economic MPC for chemical process, periodic operation is not pathological but rather constitutes the normal operation since it can offer better long-time averaged economic performance(Müller (2014)).

In Shiriaev and Fradkov (2001), periodic orbits stabilization is achieved by the Lyapunov function based the information of first integrals for the drift dynamics. A review on the periodic motion planning and feedback control design based on the Poincaré first-return map and the transverse linearization is presented in Shiriaev et al. (2008). All these approaches focus on the periodic orbits in the phase space without considering the time domain information – the trajectory tracking. In Hudon and Guay (2010), time-dependent damping tracking controller is developed based on time-varying dissipative potential with respect to the difference between the drift vector field and the reference dynamics. But there is no general rules to show how to choose a proper closed 2-form and the domain of attraction of the time-varying controller.

In this paper, tracking control of periodic operation is considered in the framework of contraction theory. System analysis and control synthesis are performed on the displacement dynamics which is the original nonlinear system equipped with its differential dynamics (continuous linearization). First, the synthesis problem with respect to control contraction metrics (CCM) and differential state feedback control is addressed, which can be seen as the "differential" version of control Lyapunov function and state feedback control. For polynomial systems, the non-convex contraction inequality can be converted into a convex sum-of-squares (SOS) optimization problem through a relaxation method, which can be further translated into a semi-definite programming by SOS softwares, such as YALMIP (Löfberg (2004)). A feasible CCM can be used to calculated the geodesics connecting the reference point and current state at each time instant. The actual control law is the integral of differential state feedback control along the geodesics. Furthermore, a trajectory dependent Lyapunov function, which reflects the closed-loop asymptotic tracking behavior, can be constructed from the CCM automatically.

In this work, the control action is the integral of differential state feedback along the geodesics connecting the reference trajectory and state trajectory. Thus, the information feedback to the system is the geodesics, a general form of deviation, which usually is a curve with respect to a Riemannian metric. The control synthesis part does not involves the reference trajectory which means that the differential feedback control is trajectory independent. Therefore, the change of periodic operation mode does not require a new control design procedure as long as the reference trajectory is feasible for the nonlinear dynamics and certain constraints for synthesis results. Moreover, since the synthesis problem is solved through SOS programming for polynomial systems, it is a flexible numerical analytic approach, which can take certain level of analytic nonlinearity into account while allowing for

⋆ This work was supported by ARC Discovery Project DP130103330. The first author acknowledges the financial support of the China Council Scholarship and the UNSW Tuition Fee Scholarship.

10.1016/j.ifacol.2016.07.404

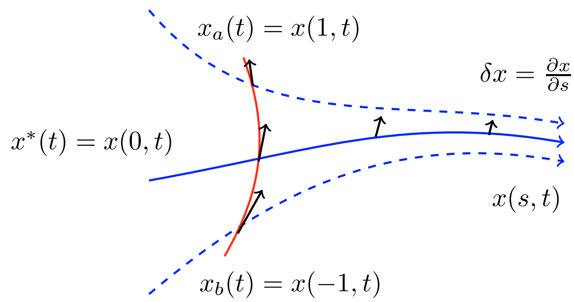

Fig. 1. Graphical interpretation of displacement dynamics

numerical optimization to search for a proper CCM and differential feedback control. Furthermore, the contraction rate and attraction domain can be specified in the SOS formulation, to meet the design specifications on tracking performance.

The remainder of this paper is structured as follows. In Section 2 we presents the background of control contraction metrics including displacement dynamics, synthesis problem and control algorithm. In Section 3, the nonconvex control synthesis task is converted into an SOS programming problem through a relaxation method. An application of the proposed approach to the Lotka-Volterra (LV) model of sustained oscillating reactions is presented in Section 4. Conclusion and future areas for investigation are outlined in Section 5.

2. PRELIMINARIES

2.1 Control synthesis based on contraction analysis

Consider following nonlinear control-affine system

$$\dot{x} = f(x) + B(x)u \tag{1}$$

where $x \in \mathcal{M}$, $u \in \mathcal{U}$ are state and control variables, $\mathcal{M}$ and $\mathcal{U}$ are state manifold and input manifold which are locally diffeomorphic to $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Contraction analysis is based on the *displacement dynamics* which is the combination of (1) and its associated differential dynamics (Lohmiller and Slotine (1998)):

$$\delta\dot{x} = A(x,u)\delta x + B(x)\delta u \tag{2}$$

where $A(x,u) = \frac{\partial}{\partial x}(f(x) + B(x)u)$ is affine in u.

Remark 1. A graphical representation of the virtual displacement $(\delta x, \delta u)$ is given by Forni and Sepulchre (2014). Consider a continuous parameterized $\mathcal{C}^2$ solution family $(x,u)(s,t)$ of (1) where $s \in \mathbb{R}$ is the implicit parameter as shown in Figure 1, by taking derivatives of (1) with respect to parameter s, we obtain

$$\frac{\partial^2 x}{\partial s \partial t} = \frac{\partial^2 x}{\partial t \partial s} = A(x,u)\frac{\partial x}{\partial s} + B(x)\frac{\partial u}{\partial s}. \tag{3}$$

Comparing it with 2, the displacement $(\delta x, \delta u)$ is nothing but the tangent vector $(\frac{\partial x}{\partial s}, \frac{\partial u}{\partial s}) \in T\mathcal{M} \times T\mathcal{U}$. So the differential dynamics (2) characterizes how tangent vector travels along any feasible state trajectory. For example, if δx vanishes along $x^*(t)$ (as shown in Figure 1), all trajectories contract to each other and converge to the reference trajectory $x^*(t)$.

The state manifold $\mathcal{M}$ is said to be a Riemannian manifold if it is endowed with a Riemannian metric tensor $g : T\mathcal{M} \times T\mathcal{M} \to \mathbb{R}^+$ where $g(\delta_1, \delta_2) = \delta_1^T M(x)\delta_2$ and Riemannian metric $M(x)$ is smooth positive definite matrix function. Given any Riemannian metric $M(x)$, the length of geodesics connecting two points x_0 and x_1 can be calculated through following optimization problem:

$$\begin{aligned} \min_{\gamma} L = & \int_0^1 \sqrt{\dot{\gamma}^T(s) M(\gamma(s))\dot{\gamma}(s)} ds \\ \textbf{s.t.} \quad & \gamma(s) \text{ is a piecewise } \mathcal{C}^1 \text{ function} \\ & \gamma(0) = x_0, \gamma(1) = x_1. \end{aligned} \tag{4}$$

From system analysis point of view, the Riemannian metric tensor is differential Lyapunov function (Forni and Sepulchre (2014)). The SOS programming is applied to search for Riemannian metric for polynomial systems in Aylward et al. (2008). The control contraction metric can be thought of as a special type of Riemannian metric which satisfies that the value of $g(\delta, \delta)$ for any displacement δ of (2) decays along the flow of system (1) with certain control action. The basic idea of CCM is given by Manchester and Slotine (2015). Suppose every state trajectory is locally stabilizable where each point $x(s,t)$ has a small region of stability, but if a "chain" of states x_k, $k = 0, 1, \ldots, n$ joining the current state $x(1,t)$ to the reference point $x(0,t)$ is stabilized, in the sense that if x_k lies in the stability region of x_{k+1}, then $x(t)$ can be driven towards $x^*(t)$ steps by steps. As the number of chain states goes to infinity, the limit of chain becomes a smooth path $\gamma(s)$, $s \in [0,1]$ in the state space. The differential dynamics (2) can also describe the dynamics of tangent vector travel along the state space. In this sense, the stability result of differential dynamics implies the contraction behavior of state trajectories. Following theorem gives out the control synthesis based on contraction analysis.

Theorem 2. (Manchester and Slotine (2014)). For a positive constant λ, if there exist a contraction metric $M(x)$ and a differential state feedback law $\delta u = K(x,u)\delta x$ where K is affine in u for displacement dynamics (1), (2) satisfying following differential Lyapunov inequality:

$$\dot{M} + (A+BK)^T M + M(A+BK) < -2\lambda M, \ \forall x \in \mathcal{M}, u \in \mathcal{U}, \tag{5}$$

then the system is universally stabilizable by following state feedback control law:

$$u(t) = u^*(t) + \int_0^1 K(\gamma(s), u(s,t))\dot{\gamma}(s) ds \tag{6}$$

where $\gamma(s)$ is the geodesics connecting $x^*(t)$ and $x(t)$ subject to the contraction metric $M(x)$. The contraction rate between closed-loop trajectory and the reference is λ, that is

$$L(x(t), x^*(t)) \leq k e^{-\lambda t} L(x(0), x^*(0)) \tag{7}$$

where k is positive constant.

Remark 3. The CCM to contraction theory is like control Lyapunov function to Lyapunov stability theory. In order to handle bilinear terms: MBK, $K^T B^T M$, Condition (5) is transformed to following inequality:

$$\dot{W} - AW - WA^T - BY - Y^T B^T - 2\lambda W > 0 \tag{8}$$

where $W = M^{-1}$ and $Y = KW$. However, Condition (8) is affine in u since following terms $\dot{W}, A, Y$ are affine in u, which is still infeasible for SOS programming. If u is treated as an implicit function of x as shown in (6), the set of W and Y satisfying (8) is not jointly convex.