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On Periodic Solutions of Singularly Perturbed Integro-differential Volterra Equations with Periodic Nonlinearities ⋆

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Abstract:
We examine nonlinear singularly perturbed systems, described by integro-differential equations with periodic nonlinearities. Equations with periodic nonlinearities govern phase-locked loops and other synchronization circuits, as well as many "pendulum-like" systems, arising in mechanics and physics. The presence of periodic nonlinearity typically endows the system with infinite sequence of equilibria points. One of the central questions related to such systems is whether any solution converges to one of the equilibria (which is sometimes referred to as the *gradient-like behavior*) or some oscillatory solutions exist. Under singular perturbation, the self-standing problem is the persistence of the gradient-like behavior as the small parameter tends to zero. In spite of substantial efforts in solving these problem, the existing conditions for the gradient-like behavior (which guarantee, in particular, the absence of oscillations) are only sufficient and may be quite conservative. In this paper we demonstrate that their relaxation guarantees inexistence of *special* oscillatory trajectories, namely, periodic solutions of high frequency. We give constructive frequency-domain conditions, which guarantee that all periodic solutions in the system, if they exist, have frequencies lower than some predefined constant. An important property of this estimate is its *uniformity* with respect to the small parameter.

Keywords: Singular perturbation, periodic solution, integro-differential equation, phase synchronization systems, Fourier series.

1. INTRODUCTION

Singularly perturbed equations are used to model wide range of natural and engineered systems, operating on two different timescales and having thus "fast" and "slow" modes. Numerous examples and main historical milestones of the singular perturbation theory can be found in Fridrichs (1955); Dyke (1964); Cole (1968); Kokotovic et al. (1986); O'Malley (1991); Naidu and Calise (2001). Singularly perturbed equations involve a small scalar parameter, whose vanishing causes changes in system's structure or other discontinuities. Typically, the order of the highest derivative or the state vector dimension decreases.

Up to now, the efforts in the mathematical analysis of singular perturbations has been mainly focused on two problems. The first problem deals with asymptotical properties of the perturbed system as the parameter tends to zero. The central result, sometimes referred to as the Tikhonov approximation theorem Tikhonov (1948), establishes convergence of the perturbed solutions to the unperturbed ones. The Tikhonov-type theorems were elaborated for a wide class of ODE systems Vasil'eva (1963) and later extended to integral equations and PDE; some recent developments can be found in Lizama and Prado (2006a,b); Parand and Rad (2011); Tang et al. (2016). The second problem is concerned with criteria for *stability* under sufficiently small parameter, pioneered by Klimushev and Krasovskii (1961) and extended to a wide class of linear Cao and Schwartz (2004) and nonlinear systems Chow (1978); Khalil (1981). The extensive studies on stability and asymptotic analysis enabled the expansion of the classical control, identification and filtering theories to singularly perturbed systems Kokotovic et al. (1986).

In spite of this substantial progress, many important problems related to the dynamics of singularly perturbed systems still remain open. For instance, unlike global asymptotical stability properties, the effects of multistability caused by presence of multiple equilibria points are almost uncovered by the existing literature on singularly perturbed systems. In this paper we consider a wide class of systems that can be represented as the feedback interconnection of a linear part and *periodic* nonlinearity. Such systems usually have infinite sequences of stable and unstable equilibria, as exemplified by the mathemat-

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ical pendulum and other pendulum-like systems Stoker (1950). Similar models describe phase-locked loops (PLL) and other synchronization circuits, arising in electrical and communication engineering Margaris (2004); Leonov et al. (2015). In view of these applications, a term "phase synchronization" systems (PSS) for models with periodic nonlinearities was coined Leonov (2006).

One of the central questions regarding the dynamics of PSS is the global attractivity of the equilibrium set, that is, convergence of any solution to one of the equilibria. This property, sometimes referred to as the *gradient-like* behavior Leonov (2006), excludes the possibility of oscillatory solutions, considered to be undesirable in electric circuits Leonov et al. (2015). Efficient sufficient conditions for the gradient-like behavior in the "frequency-algebraic" form, based on the periodic Lyapunov functions and the Popov's method Popov (1973), can be found in Leonov (2006); Leonov et al. (1996); Perkin et al. (2012); Smirnova and Proskurnikov (2016).

In the case where the gradient-like behavior of a system cannot be proved, a natural question arises which oscillatory and, in particular, periodic solutions it has. Existence of periodic solutions of some prescribed frequency in special phase locked loops was studied in Shakhil'dyan and Lyakhovkin (1972); Evtyanov and Snedkova (1968). In the later paper Leonov and Speranskaya (1985) a general *inexistence* criterion was obtained, employing the Fourier series method. It was shown that a relaxed version of the condition for the gradient-like behavior guarantees absence of "fast-oscillating" periodic solutions, whose frequencies are beyond the prescribed range. The results of Leonov and Speranskaya (1985) were extended to discrete-time Leonov and Fyodorov (2011) and infinite dimensional Leonov et al. (1996) PSS. The latter results were extended, with tightening of the frequency-algebraic conditions, in Perkin et al. (2015); Smirnova and Proskurnikov (2016) by employing novel Popov-type functionals from Perkin et al. (2012).

In this paper we extend the results of Perkin et al. (2015) to PSS modeled by *singularly perturbed* equations and get frequency-domain criteria for the inexistence of the fast-oscillating periodic solutions. These conditions are *uniform* with respect to the small parameter. It should be noticed that singular perturbations naturally arise in many models, related to mechanical and electrical systems, e.g. various relaxation oscillations O'Malley (1991). In PLL, singularly perturbed Volterra equations naturally describe the effects of "weak filtering", where the small parameter determines the filter bandwidth Hoppensteadt (1983).

2. PROBLEM SETUP

In this paper, we deal with the integro-differential Volterra equation with a small parameter μ at the higher derivative:

$$\mu\ddot{\sigma}_\mu(t) + \dot{\sigma}_\mu(t) = \alpha(t) + \rho\varphi(\sigma_\mu(t-h)) - \int_0^t \gamma(t-\tau)\varphi(\sigma_\mu(\tau))d\tau \ (t \geq 0). \tag{1}$$

Here $\mu > 0, h \geq 0, \rho \in \mathbf{R}, \gamma, \alpha\colon [0,+\infty) \to \mathbf{R}$, $\varphi\colon \mathbf{R} \to \mathbf{R}$. The map φ is assumed C^1 - smooth and Δ-periodic with two zeros on $[0,\Delta)$ and the property

$$\varphi(\sigma_\mu)^2 + \varphi'(\sigma_\mu)^2 \neq 0.$$

Without loss of generality, we assume that

$$\int_0^\Delta \varphi(\sigma)d\sigma \leq 0. \tag{2}$$

We suppose that the kernel function $\gamma(\cdot)$ is piece-wise continuous and the function $\alpha(\cdot)$ is continuous.We assume also that the linear part of (1) is stable:

$$|\alpha(t)| + |\gamma(t)| \leq Me^{-rt} \quad (M, r > 0). \tag{3}$$

For each μ the solution of (1) is uniquely defined by specifying initial condition

$$\sigma_\mu(t)|_{t\in[-h,0]} = \sigma^0(t). \tag{4}$$

Here $\sigma^0(\cdot)$ is C^1-smooth with $\sigma_\mu(0+0) = \sigma^0(0)$ and $\dot{\sigma}_\mu(0+0) = \dot{\sigma}^0(0)$. We put by definition

$$s_1 = \inf_{\sigma\in[0,\Delta)} \frac{d\varphi}{d\sigma} < 0, \quad s_2 = \sup_{\sigma\in[0,\Delta)} \frac{d\varphi}{d\sigma} > 0,$$

so that the nonlinearity satisfies the slope restrictions

$$s_1 \leq \frac{d\varphi}{d\sigma} \leq s_2, \quad \forall\sigma \in \mathbf{R}. \tag{5}$$

In this paper, we are interested in the criteria, ensuring the inexistence of periodic solutions in (1) under arbitrary sufficiently small μ. We start with a formal definition. In the theory of functional and differential equations, the periodic solutions to the equation are typically defined as periodic functions, obeying this equation. Dealing with PSS (in particular, PLL), the solution periodicity is understood in a broader sense Leonov et al. (1996). The phase $\sigma(t)$ is not necessarily periodic function but may "slip several cycles" Stoker (1950) over the period, whereas the functions $\varphi(\sigma(t))$ (the phase detector's output) and $\dot{\sigma}(t)$ ("frequency") are periodic in the usual sense.

Definition 1. We say a solution $\sigma_\mu(t)$ of (1) is periodic if there exist a number $T_\mu > 0$ and integer I_μ such that

$$\sigma_\mu(t + T_\mu) = \sigma_\mu(t) + I_\mu\Delta, \ \forall t. \tag{6}$$

If $I_\mu = 0$ the solution $\sigma_\mu(t)$ is called a periodic solution of the first kind. If $I_\mu \neq 0$ it is called a periodic solution of the second kind. The number T_μ is the period and the number $\omega_\mu = \frac{2\pi}{T_\mu}$ is the frequency of a periodic solution.

Equation (1) is a singular perturbation of the following integro-differential Volterra equation (obtained as $\mu = 0$)

$$\dot{\sigma}_0(t) = \alpha(t) + \rho\varphi(\sigma_0(t-h)) - \int_0^t \gamma(t-\tau)\varphi((\sigma_0(\tau))\,d\tau. \tag{7}$$

In the recent paper Perkin et al. (2015), the conditions for the absence of periodic solutions in systems (7) were obtained. These conditions, employing the transfer function

$$K_0(p) = -\rho e^{-hp} + \int_0^{+\infty} \gamma(t)e^{-pt}\,dt \quad (p \in \mathbf{C}), \tag{8}$$

have the form of frequency-algebraic inequalities with varying parameters. Below we extend the results of Perkin et al. (2015) to singularly perturbed equations (1).

Let us reduce equation (1) to integro-differential Volterra equation