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Differentially Controllable Subspace for Linear Periodic Systems

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Abstract: It is well known that the concepts of controllability and differential controllability are coincident for linear time-varying systems with analytic coefficients. In this note, we discuss the difference between these concepts for linear periodic continuous-time systems with piecewise-analytic coefficients. We propose the concept of differentially controllable subspace in order to compare with the controllable subspace, and then, the formulae for computing the differentially controllable subspace are derived in both integral and differential forms. Reachability and differential reachability are also discussed in a similar way. The significance of assuming piecewise-analytic coefficients for computing the differentially controllable subspace is explained in detail by a counterexample to the previous work.

Keywords: linear systems; time-varying systems; periodic structures; continuous time systems; controllability; observability; system analysis.

1. INTRODUCTION

Controllability is a fundamental concept for characterizing the effect of control input to the state variables. A state of a linear time varying system is said to be controllable if the state is transferred to the origin by an admissible control input on a finite interval. A similar but different concept is differential controllability. A state of a linear time varying system is said to be differentially controllable if the state is transferred to the origin by an admissible control input on an arbitrary small interval[1]. It is well known that those concepts are coincident when the coefficient matrices of the linear time varying system are composed of analytic functions[2]. Those concepts are still coincident when the system is classified into a constant rank system [3]. But, those concepts are not coincident when the system is classified into a piecewise constant rank system [4].

In this note, we further discuss the difference between two concepts. For simplicity, we restrict the class of coefficient matrices to be periodic and piecewise-analytic. Because the coefficient matrices are periodic, the dimension of the controllable subspace is constant with respect to time. Because the coefficient matrices are piecewise-analytic, the Taylor expansion of the coefficient matrices and the state transition matrices are well-defined almost everywhere. The concept of differentially controllable subspace is then introduced in a similar way to the controllable subspace. Hereafter, we study the properties of differentially controllable subspace and compare them with those of the controllable subspace. The reachable subspace and the differential reachable subspace are also discussed in a similar way.

We use the following notations. If the function $P(t)$ is periodic with a period $T > 0$, i.e., $P(t+T) = P(t)$ for $t \in \mathbb{R}$, it is called T-periodic. The set of all piecewise continuous functions from an interval $N \subset \mathbb{R}$ to $\mathbb{R}^{n\times m}$ is denoted by $C^{\rm pc}(N, \mathbb{R}^{n\times m})$. The set of all analytic functions from an interval $N \subset \mathbb{R}$ to $\mathbb{R}^{n\times m}$ is denoted by $C^{\omega}(N, \mathbb{R}^{n\times m})$.

2. PRELIMINARIES

Consider the linear T-periodic continuous-time system

$$\dot{x} = A(t)x + B(t)u, \quad \dot{x} := \frac{dx}{dt}, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable and $u(t) \in \mathbb{R}^m$ is the control input. The coefficient matrices $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$ are supposed to be T-periodic and piecewise-analytic, where the function is said to be analytic is the convergence radius of the Taylor series expansion is positive. $A(t)$ and $B(t)$ are supposed to be analytic on open intervals $D_i = (d_i, d_{i+1})$, where the interval $[0, T)$ is divided into $[0, T) = [d_1, d_2) \cup \cdots \cup [d_q, d_{q+1})$.

Let $\Phi(s, t)$ denote the state transition matrix of $\dot{x} = A(t)x$. Because $A(t)$ is analytic on D_i, $\Phi(s, t)$ is also analytic at $s \in D_i$ and $t \in D_i$.

Define a linear operator $L_A : C^{\omega}(D, \mathbb{R}^{n\times m}) \to C^{\omega}(D, \mathbb{R}^{n\times m})$ as follows:

$$L_A F(t) := \frac{d}{dt}F(t) - A(t)F(t), \tag{2}$$

where $A \in C^{\omega}(D, \mathbb{R}^{n\times n})$ and $F \in C^{\omega}(D, \mathbb{R}^{n\times m})$ are supposed to be analytic on an open interval $D \subset \mathbb{R}$. Define the k-th repetition of L_A as follows:

$$L_A^0 F(t) := F(t), \tag{3}$$

$$L_A^1 F(t) := L_A F(t), \tag{4}$$

$$L_A^k F(t) := L_A(L_A^{k-1} F(t)), \quad k = 2, 3, \cdots. \tag{5}$$

Define a linear subspace on D as follows:

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$$\mathcal{L}^k(t) := \operatorname{Im} B(t) + \operatorname{Im} L_A B(t) + \cdots + \operatorname{Im} L_A^k B(t), \quad k = 0, 1, \cdots, \tag{6}$$

$\mathcal{L}^k(t)$ is nondecreasing with respect to k

$$\mathcal{L}^0(t) \subset \mathcal{L}^1(t) \subset \mathcal{L}^2(t) \subset \cdots, \tag{7}$$

and $\dim \mathcal{L}^k(t)$ is also nondecreasing with respect to k. Because $\dim \mathcal{L}^k(t)$ is bounded by n, there exists a nonnegative integer k^* such that $\mathcal{L}^k(t) = \mathcal{L}^{k^*}(t), k \geq k^*$. Define $\mathcal{L}^\infty(t)$ by $\mathcal{L}^\infty(t) := \mathcal{L}^{k^*}(t)$. $\mathcal{L}^\infty(t)$ is then formally given by

$$\mathcal{L}^\infty(t) = \operatorname{Im} B(t) + \operatorname{Im} L_A B(t) + \cdots. \tag{8}$$

3. SUBSPACE

3.1 Controllable subspace

The controllable subspace $\mathcal{C}(t)$ is defined as follows [5]:

Definition 1.

$$\mathcal{C}(t) := \bigcup_{s \in [t,\infty)} \left\{ \int_t^s \Phi(t,\tau) B(\tau) u(\tau) d\tau : u \in C^{\mathrm{pc}}([t,s], \mathbb{R}^m) \right\}$$

$\mathcal{C}(t)$ satisfies the following properties:

Lemma 2. [6] (i) $\mathcal{C}(t)$ is given by

$$\mathcal{C}(t) = \operatorname{Im} W_c(t, t+nT), \ t \in \mathbb{R}, \tag{9}$$

where W_c is the controllability Gramian given by

$$W_c(t,s) := \int_t^s \Phi(t,\tau) B(\tau) B(\tau)^{\mathrm{T}} \Phi(t,\tau)^{\mathrm{T}} d\tau. \tag{10}$$

(ii) $\mathcal{C}(t)$ is Φ-invariant, i.e.,

$$\mathcal{C}(t) = \Phi(t,s)\mathcal{C}(s), \ t,s \in \mathbb{R}. \tag{11}$$

(iii) $\mathcal{C}(t)$ is T-periodic, i.e.,

$$\mathcal{C}(t) = \mathcal{C}(t+T), \ t \in \mathbb{R}. \tag{12}$$

(iv) The dimension of $\mathcal{C}(t)$ is constant, i.e.,

$$\dim \mathcal{C}(t) = \dim \mathcal{C}(0), \ t \in \mathbb{R}. \tag{13}$$

3.2 Reachable subspace

The reachable subspace $\mathcal{R}(t)$ is defined as follows [5]:

Definition 3.

$$\mathcal{R}(t) := \bigcup_{p \in (-\infty,t]} \left\{ \int_p^t \Phi(t,\tau) B(\tau) u(\tau) d\tau : u \in C^{\mathrm{pc}}([p,t], \mathbb{R}^m) \right\}$$

$\mathcal{R}(t)$ satisfies the following properties:

Lemma 4. [6] (i) $\mathcal{R}(t)$ is given by

$$\mathcal{R}(t) = \operatorname{Im} W_r(t-nT, t), \ t \in \mathbb{R}, \tag{14}$$

where W_r is the controllability Gramian given by

$$W_r(t,s) := \int_t^s \Phi(s,\tau) B(\tau) B(\tau)^{\mathrm{T}} \Phi(s,\tau)^{\mathrm{T}} d\tau. \tag{15}$$

(ii) $\mathcal{R}(t)$ is Φ-invariant, i.e.,

$$\mathcal{R}(t) = \Phi(t,s)\mathcal{R}(s), \ t,s \in \mathbb{R}. \tag{16}$$

(iii) $\mathcal{C}(R)$ is T-periodic, i.e.,

$$\mathcal{R}(t) = \mathcal{R}(t+T), \ t \in \mathbb{R}. \tag{17}$$

(iv) The dimension of $\mathcal{R}(t)$ is constant, i.e.,

$$\dim \mathcal{R}(t) = \dim \mathcal{R}(0), \ t \in \mathbb{R}. \tag{18}$$

Moreover, it is shown that $\mathcal{C}(t)$ and $\mathcal{R}(t)$ are coincident for linear periodic systems.

Lemma 5. [6]

$$\mathcal{C}(t) = \mathcal{R}(t), \ t \in \mathbb{R}. \tag{19}$$

We note that $\mathcal{C}(t)$ and $\mathcal{R}(t)$ are not coincident for linear time-varying system other than periodic ones.

3.3 Differentially controllable subspace

To investigate the instantaneous control action, we focus on the concept of differential controllability [1].

Definition 6. Given the state $x(t) = x_0$ of the system (1) at time t. The state $x_0 \in \mathbb{R}^n$ is said to be differentially controllable at time t if there exist $s \in [t,p)$ and $u \in C^{\mathrm{pc}}([t,s], \mathbb{R}^m)$ satisfying

$$\Phi(s,t)x_0 + \int_t^s \Phi(s,\tau) B(\tau) u(\tau) d\tau = 0$$

for any $p > t$. The system (1) or the (A,B)-pair is said to be differentially controllable if all states are differentially controllable at time t.

In accordance with Definition 6, we introduce the differentially controllable subspace as follows:

Definition 7. The set of all states which are differentially controllable at time t is denoted by

$$\mathcal{C}_d(t) := \bigcap_{p \in (t,+\infty)} \bigcup_{s \in [t,p)} \left\{ \int_t^s \Phi(t,\tau) B(\tau) u(\tau) d\tau : u \in C^{\mathrm{pc}}([t,s], \mathbb{R}^m) \right\},$$

and is called the differentially controllable subspace at time t.

As shown in Eq. (9), $\mathcal{C}(t)$ is given by the image of $W_c(t, t+nT)$ defined on a finite interval $(t, t+nT)$. In contrast, $\mathcal{C}_d(t)$ is given by the image of $W_c(t,s)$ defined on an infinitesimal interval (t,s), where $s > t$ is sufficiently close to t. By taking the orthogonal complement of $\mathcal{C}_d(t)$, $\mathcal{C}_d(t)^\perp$ is computed by using $\Phi(s,t)$ and $B(t)$ as follows:

Proposition 8.

$$\begin{aligned}\mathcal{C}_d(t)^\perp &= \bigcap_{p \in (t,+\infty)} \bigcup_{s \in [t,p)} \operatorname{Ker} W_c(t,s) \\ &= \bigcap_{p \in (t,+\infty)} \bigcup_{s \in [t,p)} \left\{ \xi \in \mathbb{R}^n : \xi^{\mathrm{T}} \Phi(t,\tau) B(\tau) = 0, \tau \in (t,s) \right\}. \end{aligned} \tag{20}$$

Proof: In a similar way to compute $\mathcal{C}(t)$, $\mathcal{C}_d(t)$ is given by

$$\mathcal{C}_d(t) = \bigcap_{p \in (t,+\infty)} \bigcup_{s \in [t,p)} \operatorname{Im} W_c(t,s).$$

Because $W_c(t,s)$ is symmetric, the first equal sign is obtained. Because $\xi^{\mathrm{T}} W_c(t,s) = 0$ and $\xi^{\mathrm{T}} \Phi(t,\tau) B(\tau) = 0, \tau \in (t,s)$ are equivalent, the second equal sign is obtained. □

We note that Eq. (20) requires integral calculations of $\Phi(t,s)$ and $W_c(t,s)$ from $A(t)$ and $B(t)$, and therefore, Eq. (20) can be taken as the integral form.

Equation (20) is valid for all $t \in \mathbb{R}$. We obtain another formula which is only valid on D_i as follows: