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Norm saturating property of time optimal controls for wave-type equations ⋆

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Abstract: We consider a time optimal control problem with point target for a class of infinite dimensional systems governed by abstract wave operators. In order to ensure the existence of a time optimal control, we consider controls of energy bounded by a prescribed constant $E > 0$. Even when this control constraint is absent, in many situations, due to the hyperbolicity of the system under consideration, a target point cannot be reached in arbitrarily small time and there exists a minimal universal controllability time $T_* > 0$, so that for every points y_0 and y_1 and every time $T > T_*$, there exists a control steering y_0 to y_1 in time T. Simultaneously this may be impossible if $T < T_*$ for some particular choices of y_0 and y_1 .

In this note we point out the impact of the strict positivity of the minimal time T_* on the structure of the norm of time optimal controls. In other words, the question we address is the following: If τ is the minimal time, what is the L^2-norm of the associated time optimal control? For different values of y_0, y_1 and E, we can have $\tau \leqslant T_*$ or $\tau > T_*$. If $\tau > T_*$, the time optimal control is unique, given by an adjoint problem and its L^2-norm is E, in the classical sense. In this case, the time optimal control is also a norm optimal control. But when $\tau < T_*$, we show, analyzing the string equation with Dirichlet boundary control, that, surprisingly, there exist time optimal controls which are not of maximal norm E.

Keywords: Wave equations, Optimal control, Open loop control systems, Point-to-point control, Reachable states, Norm-optimal controls, Minimal control time.

1. INTRODUCTION

Time optimal control problems have been intensively studied for finite dimensional systems showing that the optimal control satisfies a Pontryagin maximum principle, it is bang-bang and unique. For a survey of these results, we refer to the books Lee and Markus (1967) and Agrachev and Sachkov (2004) and to the original work by Bellman et al. (1956). These results have been extended in Fattorini (1964) to infinite dimensional systems and reported in the books by Lions (1968) and Fattorini (2005).

Many new results have been obtained for parabolic type systems; see for instance Mizel and Seidman (1997), Wang (2008), Phung and Wang (2013) and Kunisch and Wang (2013). However only few results exist for conservative systems and they only concern distributed controls; see for instance Fattorini (1977), Lohéac and Tucsnak (2013) and Kunisch and Wachsmuth (2013b,a).

In all the above mentioned works, in order to ensure the existence of a time optimal control, the controls are assumed to be bounded in L^∞. But for the wave equation, due to the finite velocity of propagation, the main difficulty arises from the fact that it is globally controllable only for a large enough control time.

In the present work, in order to analyse this delicate issue, we chose an Hilbertian approach and assume that the control is bounded in L^2. This simplification allows us to easily consider the case of boundary control operators. In section 4, we consider the example of the string equation with Dirichlet boundary control, where some computations are explicit.

According to Gugat and Leugering (2008) (Theorem 3.1), the string equation with Dirichlet boundary control cannot be controlled with classical bang-bang controls, i.e. controls taking their values in $\{-1, 1\}$ for almost every time. In addition, for norm optimal control problems, which is a problem related to the one of finding time optimal controls as we will see later, Bennighof and Boucher (1992) consider a string equation with Newman control at both ends and prove that for constant state targets (with constant initial data), the time optimal controls are of bang-off-bang type, i.e. controls taking values in $\{-1, 0, 1\}$ for almost every time. Thus, even for constant data, time optimal controls are not, in general, of bang-bang form. A more general result on L^∞-norm optimal controls, for the same system, can be found in Gugat (2002) and its generalisation to any L^p-norm optimal controls in Gugat and Leugering (2002).

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In order to give a precise statement of our result, let us first recall some classical definitions and notations from control theory, see for instance Tucsnak and Weiss (2009).

Throughout this paper, X and U are real Hilbert spaces identified with their duals. We denote by $\mathbb{T} = (\mathbb{T}_t)_{t\geqslant 0}$ a strongly continuous semigroup on X generated by an operator $A : \mathcal{D}(A) \to X$. In all this paper, we assume that A is skew-adjoint with nonempty resolvent $\rho(A)$.
The notation X_1 stands for $\mathcal{D}(A)$ equipped with the norm $\|z\|_{X_1} = \|(\beta \mathrm{Id} - A)z\|_X$, where $\beta \in \rho(A)$ is fixed, while X_{-1} is the completion of X with respect to the norm $\|z\|_{X_{-1}} = \|(\beta \mathrm{Id} - A)^{-1}z\|_X$. Let us also denote by A and $\mathbb{T}$ the extensions of A to X and $\mathbb{T}$ to X_{-1}.
Let us now introduce the control operator $B \in \mathcal{L}(U, X_{-1})$. Then the infinite dimensional system under consideration is:

$$y' = Ay + Bu \qquad y(0) = \mathrm{y}_0\,, \tag{1}$$

where y is the state of the system and $u \in L^2(\mathbb{R}_+, U)$ is the control and $\mathrm{y}_0 \in X$ is the initial state position. The solution of (1) is:

$$y(t) = \mathbb{T}_t \mathrm{y}_0 + \Phi_t u \qquad (t \geqslant 0)\,,$$

where $\Phi_t \in \mathcal{L}\left(L^2([0,t], U), X_{-1}\right)$ is the input to state map defined by:

$$\Phi_t u = \int_0^t \mathbb{T}_{t-s} Bu(s)\,\mathrm{d}s \qquad (t \geqslant 0\,,\ u \in L^2([0,t], U))\,.$$

We will say that B is an *admissible control operator for* $\mathbb{T}$ if there exists $t > 0$ such that $\operatorname{Ran}\Phi_t \subset X$ and in the sequel we will assume that the pair $(\mathbb{T}, B)$ satisfies this condition. Finally, we will say that the pair (A, B) is *exactly controllable in time* T $(T > 0)$ if $\operatorname{Ran}\Phi_T = X$. In the sequel, we will assume that the pair (A, B) is exactly controllable in some time $T > 0$ and we define the *universal controllability time*:

$$T_* = \inf\{T > 0\,,\ \operatorname{Ran}\Phi_T = X\} \geqslant 0\,. \tag{2}$$

To be more precise, the time optimal control problem we address in this work is the following:

Problem 1. Given $E > 0$ and $\mathrm{y}_0, \mathrm{y}_1 \in X$ with $\mathrm{y}_0 \neq \mathrm{y}_1$, find the minimal time $T > 0$ such that there exists $u \in L^2([0,T], U)$ satisfying:

- $\|u\|_{L^2([0,T],U)} \leqslant E$;
- the solution y of (1) with control u and initial condition y_0 satisfies $y(T) = \mathrm{y}_1$.

In all this note, E defines a given nonnegative constant.

Our first result is as follows:

Theorem 1.1. Let $\mathrm{y}_0, \mathrm{y}_1 \in X$ with $\mathrm{y}_0 \neq \mathrm{y}_1$.
Assume that the pair (A, B) is exactly controllable and fix $T > 0$. Assume that a control $u \in L^2([0,T], U)$ with $\|u\|_{L^2([0,T],U)} \leqslant E$ steering y_0 to y_1 in time T exists.

Then there exists a minimal time $\tau > 0$ such that y_0 can be steered to y_1 in time $\tau = \tau(\mathrm{y}_0, \mathrm{y}_1; E)$ preserving this bound, i.e.

$$\begin{aligned}\tau = \min\big\{T > 0\,,\ \exists u \in L^2([0,T], U)\,,\\ \|u\|_{L^2([0,T],U)} \leqslant E \quad \text{and}\\ \Phi_T u = \mathrm{y}_1 - \mathbb{T}_T \mathrm{y}_0\big\}\,. \end{aligned} \tag{3}$$

Moreover, if $\tau > T_*$ (with $T_* \geqslant 0$ defined by (2)), there exists a unique control $u \in L^2([0,\tau], U)$ with $\|u\|_{L^2([0,\tau],U)} \leqslant E$ steering y_0 to y_1 in time τ. In addition, we have

$$\|u\|_{L^2([0,\tau],U)} = E \tag{4}$$

and there exists $\eta \in X \setminus \{0\}$ such that:

$$u = \Phi_\tau^* \eta\,. \tag{5}$$

Let us remind that for every $\eta \in X$, $(\Phi_T^*\eta)(t) = B^*z(t)$ $(t \in [0,T])$ where z is solution of:

$$z' = -A^* z\,, \qquad z(T) = \eta\,.$$

The proof of the characterization of the optimal control Theorem 1.1 when $\tau > T_*$ is similar to the one by (Lohéac and Tucsnak, 2013, Theorem 1.4) and is not repeated here. We only give the key argument for the existence of τ in section 2, see Proposition 2.1.

Remark 1.1. If $\tau > T_*$ the minimal time control is the minimal norm control in time τ steering y_0 to y_1. That is to say that, if $\tau > T_*$, the time optimal control is $L^2([0,\tau], U)$-norm optimal. This fact gives the same result as the one in Wang and Zuazua (2012) for the heat equation, where we have $T_* = 0$.

Theorem 1.1 does not give any relevant information when $\tau \leqslant T_*$. In fact when $\tau \leqslant T_*$ the situation is less clear. In section 3, we show in Proposition 3.1, under suitable assumptions on the reachable set, that for $\tau < T_*$ there exists a time optimal control $u \in L^2([0,\tau], U)$ with $\|u\|_{L^2([0,\tau],U)} < E$. That is to say that, when $\tau < T_*$, there exist time optimal controls which do not satisfy the norm saturating property (4). This situation appears at least when $\mathrm{y}_0 = 0$ and it is a consequence of the following two properties of the reachable sets: They are closed and strictly increasing as a function of $\bar{t} < T_*$.

More precisely, the way we build a non saturation time optimal control is by choosing a target $\mathrm{y}_1 \in X$ so that y_1 is accessible from 0 in a time $\bar{t} > 0$ but not for times $s < \bar{t}$. In this case, it is clear that $\tau(0, \mathrm{y}_1; E) \geqslant \bar{t}$ whatever $E > 0$ is. Choosing such a target y_1 and choosing a constant $E > 0$ large enough, we will obtain that $\tau(0, \mathrm{y}_1; E) = \bar{t}$ and the existence of a time optimal control whose norm is not E. In section 4, we will show that the assumptions made in Proposition 3.1 are fulfilled for the string equation with Dirichlet boundary control.

2. WELL POSEDNESS

In this paragraph, we will prove that τ defined by (3) exists, i.e., the set

$$\begin{aligned}\big\{T > 0\,,\ \exists u \in L^2([0,T], U)\,,\ \|u\|_{L^2([0,T],U)} \leqslant E\\ \text{and} \quad \mathrm{y}_1 - \mathbb{T}_T \mathrm{y}_0 = \Phi_T u\big\}\end{aligned}$$

admits a minimum.

Before going further, let us introduce some ad hoc notations and spaces. Let us define the set of points which can be reached from 0,

$$R_t^2 = \Phi_t\left(L^2([0,t], U)\right) \qquad (t > 0)\,, \tag{6}$$

with the convention $R_0^2 = \{0\}$.
Endowed with the norm:

$$\begin{aligned}\|\mathrm{y}\|_{R_t^2} = \inf\big\{\|u\|_{L^2([0,T],U)}\,,\ u \in L^2([0,T], U)\,,\\ \mathrm{y} = \Phi_t u\big\}\,,\end{aligned}$$