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Stability analysis of a system coupled to a transport equation using integral inequalities ⋆

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Abstract: We address the stability of a system of ordinary differential equations coupled with a transport partial differential equation, using a Lyapunov functional approach. This system can also be interpreted as a finite dimensional system subject to a state delay. Inspired from recent developments on time-delay systems, a novel method to assess stability of such a class of coupled systems is developed here. We will specifically take advantage of a polynomial approximation of the infinite dimensional state of the transport equation together with efficient integral inequalities in order to study the stability of the infinite dimensional system. The main result of this paper provides exponential stability conditions for the whole coupled system expressed in terms of linear matrix inequalities and the results are tested on academic examples.

Keywords: Transport equation, Lyapunov, integral inequalities, polynomial approximation.

1. INTRODUCTION

Distributed parameter systems represent a wide class of control systems whose state is of infinite dimension. This class of system appears in numerous applications that we will not list here. Analyzing and controlling distributed parameter systems represents an attractive area of research in applied mathematics and more recently in automatic control: see for instance Prieur (2008) Susto and Krstic (2010), Smyshlyaev et al. (2010), Smyshlyaev and Krstic (2005), Krstic et al. (2009) among many others.

We study in this document the particular situation where a finite dimensional system is coupled to a transport equation, and the main difficulty in the stability analysis we will perform is related to the infinite dimensional nature of the transport part of the whole state.

It is worth noting that the class of systems we study can be also interpreted as Time Delay Systems (TDS), which have been widely investigated in the literature (see Fridman (2014); Niculescu (2001); Gu et al. (2003)). The aim of this paper is to take advantage of some recent developments on the stability analysis of TDS in order to provide a new framework for the analysis of this system of Ordinary Differential Equations (ODEs) coupled with a transport Partial Differential Equation (PDE). The first difficulty arises from the fact that stability of TDS can be assessed using the Lyapunov-Krasovskii Theorem and analyzing the stability of our system cannot be performed using exactly the same theorem. The second difficulty lies in the infinite dimensional part of the system, which prevents from extending directly the existing methods from the finite dimension analysis. In order to provide efficient stability conditions, we will construct a Lyapunov functional by enriching the classical energy of the whole system with terms built on a polynomial approximation of the infinite dimensional state expressed using Legendre polynomials.

While polynomial approximation methods for the analysis of infinite dimension systems is not a new idea (see for instance the convex optimization and sum-of squares frameworks developed in Papachristodoulou and Peet (2006); Peet (2014) or Ahmadi et al. (2014)), the novelty of this approach relies on the use of efficient integral inequalities that are able to give a measure of the conservatism associated to the approximation. These inequalities can be interpreted as a Bessel inequality on Hilbert spaces. In previous work, e.g. Seuret and Gouaisbaut (2014, 2015), the efficiency of these inequalities for the stability analysis of TDS has been shown. Indeed, one can also read in Seuret et al. (2015) a method based on a polynomial approximation of the distributed nature of the delay, using Legendre polynomials and their properties to construct Lyapunov-Krasovskii functionals. In the present paper, where a simple transport equation replaces the delay terms (an approach also studied in e.g. Bekiaris-Liberis and Krstic (2013)), an alternative use of this new method is proposed.

In the framework of the stability analysis, the present article can be seen as a first step towards the study of more intricate PDE systems using tools inherited from TDS approaches.

Notations: Herein, $\mathbb{N}$ is the set of positive integer, $\mathbb{R}_+$ the set of non-negative reals, $\mathbb{R}^n$ the n-dimensional Euclidian space with vector norm $|\cdot|_n$ and $\mathbb{R}^{n\times m}$ the set of all $n\times m$ real matrices. If $P \in \mathbb{R}^{n\times n}$ is symmetric (i.e. $P \in \mathcal{S}_n$) and positive definite, we note either $P \succ 0$ or $P \in \mathcal{S}_n^+$ and we denote by $\lambda_{\max}(P) > 0$ its largest eigenvalue. The symmetric matrix $\left[\begin{smallmatrix} A & B \\ * & C \end{smallmatrix}\right]$ stands for $\left[\begin{smallmatrix} A & B \\ B^\top & C \end{smallmatrix}\right]$ and $diag(A, B)$ is the diagonal matrix $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$. Moreover, for $A \in \mathbb{R}^{n\times n}$, we

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define $\mathrm{He}(A) = A + A^\top$. The matrix I is the identity matrix and $0_{n,m}$ stands for the matrix in $\mathbb{R}^{n\times m}$ whose entries are zero. When no confusion is possible, the subscript will be omitted. For given vectors $\{u_k\}_{k=0,\dots,n}$, we denote $\mathrm{Vect}_{k=0..n} u_k$ the vector $(u_0, \dots, u_n)^\top$. Finally, $L^2(0,1;\mathbb{R}^n)$ will denote the space of square integrable functions over the interval $(0,1) \subset \mathbb{R}$ with values in $\mathbb{R}^n$.

2. FORMULATION OF THE PROBLEM

Let us consider the following coupling of a finite dimensional system in the variable X with a transport partial differential equation in the variable z, in such a way that the transport mimics a delay term in the ODE in X:

$$\begin{cases} \dot{X}(t) = AX(t) + Bz(1,t) & t > 0, \\ \partial_t z(x,t) + \rho \partial_x z(x,t) = 0, & x \in (0,1), t > 0, \\ z(0,t) = CX(t), & t > 0. \end{cases} \tag{1}$$

The pair $(X(t), z(t)) \in \mathbb{R}^n \times L^2(0,1;\mathbb{R}^m)$ is the state of the system and it satisfies compatible initial datum $(X(0), z(x,0)) = (X^0, z^0(x))$ for $x \in (0,1)$. The matrices A, B and C are constant in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times m}$ and $\mathbb{R}^{m\times n}$.

One should know that equation $\partial_t z + \rho \partial_x z = 0$ in (1) of unknown $z = z(x,t)$ is a simple vectorial transport PDE and if the initial data $z^0 \in L^2(0,1;\mathbb{R}^m)$ and the lateral boundary data $z(0,\cdot) = CX \in L^2(\mathbb{R}_+;\mathbb{R}^m)$ are given, it has a unique solution $z \in C(\mathbb{R}_+; L^2(0,1;\mathbb{R}^m))$ such that (see e.g. Coron (2007)), for all $t > 0$:

$$\|z(t)\|_{L^2(0,1;\mathbb{R}^m)} \le \|z^0\|_{L^2(0,1;\mathbb{R}^m)} + \|X\|_{L^2(\mathbb{R}_+;\mathbb{R}^m)}.$$

The stability of System (1) will be studied thanks to a Lyapunov functional constructed with the state (X, z) and the smart use of the projection of z over the set of polynomials of degree less than a prescribed integer $N \ge 0$. We would like to emphasize that this study aims at guaranteeing the stability of the whole system through tractable LMI tests.

2.1 Legendre Polynomials and their properties

In order to express the polynomial approximation of the infinite dimensional state z, we will work with the shifted Legendre polynomials considered over the interval $[0,1]$ and denoted $\{L_k\}_{k\in\mathbb{N}}$. The main motivation for selecting these polynomials arise from their useful properties which will be described below. Instead of giving their explicit formula, we detail here their principal constitutive properties. To begin with, the family $\{L_k\}_{k\in\mathbb{N}}$ forms an orthogonal basis of $L^2(0,1;\mathbb{R})$ since

$$\langle L_j, L_k\rangle = \int_0^1 L_j(x) L_k(x) dx = \frac{1}{2k+1}\delta_{jk},$$

where δ_{jk} denotes the Kronecker delta, equal to 1 if $j = k$ and to 0 otherwise. We denote the corresponding norm of this inner scalar product $\|L_k\| = \sqrt{\langle L_k, L_k\rangle} = 1/\sqrt{2k+1}$. The boundary values are given by:

$$L_k(0) = (-1)^k, \qquad L_k(1) = 1. \tag{2}$$

Furthermore, the following derivation formula holds:

$$L'_k(x) = \begin{cases} 0, & k = 0, \\ \displaystyle\sum_{j=0}^{k-1} (2j+1)(1-(-1)^{k+j}) L_j(x), & k \ge 1. \end{cases} \tag{3}$$

One can find details about Legendre polynomials in the book by Courant and Hilbert (1989).

Therefore, any element $y \in L^2(0,1;\mathbb{R})$ can be written $y(x) = \sum_{k\ge 0} \langle y, L_k\rangle L_k(x)/\|L_k\|^2$ and throughout the paper, we will denote abusively, for $z \in C(\mathbb{R}_+; L^2(0,1;\mathbb{R}^m))$,

$$z(x,t) = \sum_{k\ge 0} \langle z(t), L_k\rangle \frac{L_k(x)}{\|L_k\|^2}$$

instead of

$$z(x,t) = \begin{bmatrix} z_1(x,t) \\ \vdots \\ z_m(x,t) \end{bmatrix} = \begin{bmatrix} \sum_{k\ge 0} \langle z_1(t), L_k\rangle L_k(x)/\|L_k\|^2 \\ \vdots \\ \sum_{k\ge 0} \langle z_m(t), L_k\rangle L_k(x)/\|L_k\|^2 \end{bmatrix}.$$

The following property will be useful hereafter.

Property 1. Let $z \in C(\mathbb{R}_+; L^2(0,1;\mathbb{R}^m))$ satisfy the transport equation in (1). The time derivative formulas

$$\begin{aligned} \frac{d}{dt}\langle z(t), L_0\rangle &= \rho z(0,t) - \rho z(1,t) \\ \frac{d}{dt}\langle z(t), L_k\rangle &= \rho \sum_{j=0}^{k-1} (2j+1)(1-(-1)^{k+j}) \langle z(t), L_j\rangle \\ &\quad + \rho(-1)^k z(0,t) - \rho z(1,t), \quad \forall k \in \mathbb{N}^* \end{aligned} \tag{4}$$

holds if $\partial_t z \in C(\mathbb{R}_+; L^2(0,1;\mathbb{R}^m))$.

The proof derives from the formulas (2) and (3).

2.2 Bessel-Legendre Inequality

The use of an approximation of the infinite dimensional state z (by a finite dimensional one using polynomials) will be efficient if we are able to measure the approximation error. The following lemma provides this kind of information.

Lemma 1. Let $z \in C(\mathbb{R}_+; L^2(0,1;\mathbb{R}^m))$ and $R \in \mathcal{S}_m^+$. The integral inequality

$$\int_0^1 z^\top(x,t) R z(x,t) dx \ge \sum_{k=0}^{N} (2k+1) \langle z(t), L_k\rangle^\top R \langle z(t), L_k\rangle \tag{5}$$

holds for all $N \in \mathbb{N}$.

Proof : It relies on the orthogonality of the Legendre polynomials and on the Bessel inequality, see e.g. Seuret and Gouaisbaut (2015). More precisely, the proof of this lemma results from the positive definiteness and the expansion of

$$\langle y_N(t), R y_N(t)\rangle = \int_0^1 y_N^\top(x,t) R y_N(x,t) dx$$

where

$$y_N(x,t) = z(x,t) - \sum_{k=0}^{N} \left\langle z(t), \frac{L_k}{\|L_k\|}\right\rangle \frac{L_k(x)}{\|L_k\|}$$

is the approximation error between the state z and its projection over the $N+1$ first Legendre polynomials. □

3. STABILITY ANALYSIS

3.1 Lyapunov-Krasovskii functional

Our objective is to construct a Lyapunov functional in order to narrow the proof of the stability of the complete