Automatica 97 (2018) 18–26

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

A new condition for asymptotic consensus over switching graphs[☆]

Nilanjan Roy Chowdhury[a,*], Srikant Sukumar[b], Debasish Chatterjee[b]

[a] Department of Electrical & Computer Engineering, North Carolina State University, Raleigh, NC 27606, USA
[b] Systems & Control Engineering, Indian Institute of Technology Bombay, Mumbai 400076, India

ARTICLE INFO

Article history:
Received 9 December 2016
Received in revised form 17 January 2018
Accepted 18 July 2018

Keywords:
Multiagent systems
Asymptotic consensus
Switched systems

ABSTRACT

We investigate asymptotic consensus of linear systems under a class of switching communication graphs. We significantly relax several reciprocity and connectivity assumptions prevalent in the consensus literature by employing switched-systems techniques to establish consensus. Our results rely solely on asymptotic properties of the switching communication graphs in contrast to classical average dwell-time conditions. A bound on the uniform rate of convergence to consensus is also established as part of this work.

1. Introduction

In the context of multi-agent systems, a group of agents is said to reach consensus when all individuals converge towards a common value/state (Ren & Beard, 2008, Page 26). Several recent results illustrate sufficient conditions guaranteeing consensus over switching graphs. Given a finite set of static graphs, the existence of a (directed) spanning tree is necessary and sufficient to guarantee consensus (Ren & Beard, 2008, Theorem 2.8) while convergence over switching graphs typically relies on the dwell-time assumption (Jadbabaie, Lin, & Morse, 2003, Theorem 2); (Olfati-Saber & Murray, 2004, Theorem 9). As documented below, more recent results on convergence analysis to consensus depend on symmetry or reciprocity conditions of the interaction graphs which restrict to a particular subclass of interactions. The current article relaxes aforementioned reciprocity conditions, and verifies asymptotic consensus under weaker assumptions for a general class of dynamic interactions.

Significant contributions to the study of consensus over dynamic graphs have been made by Jadbabaie et al. (2003), Moreau (2004) and Ren and Beard (2005) among others. The key results presented in Jadbabaie et al. (2003), Moreau (2004) and Ren and Beard (2005) however rely on the notion of *dwell-time*. For instance, a set of agents is guaranteed to reach consensus if there exists a spanning tree in each bounded consecutive time interval. A discrete-time counterpart of earlier results was proposed in Blondel, Hendrickx, Olshevsky, Tsitsiklis, et al. (2005, Theorem 1), where the authors prove consensus whenever the graph network satisfies the *ultimate connectivity* assumption: for all time $t \geq 0$, the union of the interaction graphs over the time interval $[t, \infty)$ should be connected. In the continuous-time context, recent work by Hendrickx and Tsitsiklis (2013), shows that a group of agents converge to a common limit under the assumption of *cut-balance interactions*. Cut-balance requires that for any subgroup of agents S, the cumulative in-degree and the cumulative out-degree of interactions must have a finite fixed bound on ratio. However Hendrickx and Tsitsiklis (2013) does not provide a convergence rate estimate to consensus. Following the analysis in Hendrickx and Tsitsiklis (2013), convergence to consensus is verified under weaker assumptions by Martin and Girard (2013). The authors of Martin and Girard (2013) introduce two assumptions namely, *persistent connectivity* (Martin & Girard, 2013, Assumption 1) of the interaction graphs and *slow divergence of reciprocal interaction weights* (Martin & Girard, 2013, Assumption 2) which can be considered to be weaker versions of the ultimate connectivity and the cut-balance assumptions as stated in Blondel et al. (2005) and Hendrickx and Tsitsiklis (2013) respectively. Given a class of quasi-strongly connected digraphs, convergence towards the consensus value along with measurable convergence rate was proposed in Shi and Johansson (2013). However, the analysis given in Shi and Johansson (2013) depends on a stringent *arc-balance* assumption. In addition, given a class of bicolored quasi-strongly connected digraphs, necessary and sufficient condition to global consensus is verified over nonlinear agent dynamics in Manfredi and Angeli (2017). Convergence rate to consensus for continuous-time single and double integrator systems under persistent connectivity was

☆ D. Chatterjee was supported, in part, by the grant 14ISROC010 from the Indian Space Research Organization, India. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Shreyas Sundaram under the direction of Editor Christos G. Cassandras.

* Corresponding author.
E-mail addresses: nroycho@ncsu.edu (N.R. Chowdhury), srikant@sc.iitb.ac.in (S. Sukumar), dchatter@iitb.ac.in (D. Chatterjee).

https://doi.org/10.1016/j.automatica.2018.07.018
0005-1098/

also evaluated in Chowdhury, Sukumar, and Balachandran (2016). More recently, a weaker variation of the persistent connectivity assumption stated in Martin and Girard (2013), was also proposed in Martin and Hendrickx (2016) for a class of interaction graphs where reciprocity is not instantaneous but occurs on average over time. However, the analysis proposed in Martin and Hendrickx (2016) does not lead to a computable convergence rate.

All of the aforementioned results however, pertain to single/double integrator agent dynamics. Consensus/synchronization results for agents with linear system dynamics of the form

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad \forall i \in \{1, 2, \dots, N\}, \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, have been motivated by recent contributions from Scardovi and Sepulchre (2009) and Wieland, Sepulchre, and Allgöwer (2011) that prove synchronization of all agent trajectories to a common open-loop system solution. However, each unforced agent dynamics is assumed to be at worst neutrally stable. Consensus results for agent dynamics of the form (1) that relax the 'neutrally stable' assumption were proposed in Zhang, Lewis, and Das (2011) for static digraphs and in Wen, Duan, Ren, and Chen (2014) for switching digraphs. In Wen et al. (2014) and Zhang et al. (2011) the authors bring to attention an output feedback which ensures consensus, with the control gain being a solution of an algebraic Riccati equation. However, an additional scalar gain is required for both static and dynamic graphs which depends on centralized information, specifically, the second smallest eigenvalue of the graph Laplacian (see Zhang et al., 2011, Theorem 1 for detail). Furthermore, the results in Wen et al. (2014) rely on the classical dwell-time assumption as stated in Jadbabaie et al. (2003). Li, Wen, Duan, and Ren (2015) have later shown that the notion of a centralized coupling gain can be weakened while still achieving consensus for general linear agent dynamics; the authors proposed a distributed time-varying coupling gain depending only on local information in the feedback path to achieve consensus. The results in Li et al. (2015) are, however, restricted only to static digraphs.

Some recent work towards employing switched-system results to consensus problems appear in Vengertsev, Kim, Seo, and Shim (2015), Wang and Yang (2015) and Zhai and Huang (2015). A dwell-time based consensus analysis over directed graphs was proposed in Zhai and Huang (2015). Wang et al. have later shown in Wang and Yang (2015), that the dwell-time assumption can be weakened to an average dwell-time based switching strategy. However, the results in Wang and Yang (2015) rely on the existence of connected graphs in each contiguous time-interval, which can be considered as a stronger variation of the *joint connectivity* assumption (union graph contains spanning tree) stated in the switching graph literature. This assumption was recently relaxed in Vengertsev et al. (2015) where the authors characterized a class of switching signals based on the average dwell-time strategy: a set of sufficient conditions were proposed to ensure convergence to consensus with an associated rate. An average dwell time based switched-system analysis was also proposed in Casadei, Marconi, and Isidori (2014), where the synchronization problem for a class of nonlinear homogeneous agents interacting over directed switching graphs was studied. However, the issue of computable convergence rate was not addressed in their results. The framework proposed in our article, documented in Theorems 8 and 15, is general enough to tackle nonlinear systems; see Kundu and Chatterjee (2015, 2017) for details, even though we adhere to the simpler linear setting in order not to blur the message of our work.

Our contributions are as follows:

- We identify **switching signals** to ensure asymptotic consensus over switching graphs. We relax several reciprocity and dwell-time conditions stated in Hendrickx and Tsitsiklis (2013), Martin and Girard (2013), Shi and Johansson (2013) and Jadbabaie et al. (2003), Moreau (2004) and Zhai and Huang (2015) to prove consensus under weaker assumptions. Our analysis solely relies on certain asymptotic properties of the switching signal: the frequency of switching, the fraction of activity of the constituent systems, and the density of the admissible transitions among them (see Section 2.3 for details). No dwell or average dwell-time properties on the switching signals are assumed. A novel analysis approach allows us to establish sufficient conditions for consensus. We assume general linear agent dynamics (1) for each agent with a coupling gain depending on centralized information as proposed in Wen et al. (2014) and Zhang et al. (2011).
- The current work also establishes a bound on the **uniform convergence rate** to consensus over switching graphs. Although several techniques have been utilized to analyze consensus of general linear agents over switching graphs (see e.g., Wang & Yang, 2015; Wen et al., 2014; Zhai & Huang, 2015), explicit convergence rate estimates are rare to the best of our knowledge.

Notation: The following notations are used throughout this article. For a real-valued function $f(\cdot)$, we define $\hat{f}(\cdot) := \limsup f(\cdot)$ and $\check{f}(\cdot) := \liminf f(\cdot)$. Given a matrix $M \in \mathbb{R}^{n\times n}$, $\lambda_{\max}(M)$, $\lambda_{\min}(M)$, $\lambda_2(M)$ and, $\mathrm{spec}(M)$ denote the maximum, minimum, second smallest eigenvalue and the spectrum of M. For a complex number $c \in \mathbb{C}$, $\Re(c)$ denotes the real part of c. $I_N \in \mathbb{R}^{N\times N}$ and $\mathbf{1}_N \in \mathbb{R}^N$ denote the identity matrix of dimension N and N dimensional vector containing 1 in every entry respectively. Given two positive integers n and k, $\binom{n}{k} := \frac{n!}{k!(n-k)!}$. Furthermore, given two real numbers $a, b \in \mathbb{R}$, we let $]a, b] := \{x \in \mathbb{R} : a < x \le b\}$.

Graph theory basics: We recall, primarily from Chowdhury et al. (2016) and Shi and Johansson (2013), a few notions from graph theory. A directed graph (or say digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ contains a finite non-empty node set $\mathcal{V} = \{1, 2, \dots, N\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. A relation $e_{ij} := (i, j) \in \mathcal{E}$ defines the interaction between the agents. A path of length p in graph $\mathcal{G}$ is given by a sequence of distinct arcs $v_{i_0}, v_{i_1}, \dots, v_{i_p}$ such that for each $k = \{0, 1, \dots (p-1)\}$, $(v_{i_k}, v_{i_{k+1}}) \in \mathcal{E}$. A digraph $\mathcal{G}$ is strongly connected if it contains a (directed) path for every pair of nodes $i, j \in \mathcal{V}$; $\mathcal{G}$ is quasi-strongly connected if it has a root node $v_r \in \mathcal{V}$, such that for every other vertex $v \in \mathcal{V}$ there is a directed path from v_r to v. $L_{\mathcal{G}} \in \mathbb{R}^{N\times N}$ represents the graph Laplacian matrix of a directed graph $\mathcal{G}$. $L_{\mathcal{G}}$ has at least one zero eigenvalue and all non-zero eigenvalues have positive real parts. Furthermore, $L_{\mathcal{G}}$ contains exactly one zero eigenvalue if and only if the digraph $\mathcal{G}$ is strongly or quasi-strongly connected (Ren & Beard, 2008, Corollary 2.5).

2. Problem setup

2.1. System description

This section considers a class of continuous time multi-agent system. The ith agent dynamics is modeled as

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad x_i(0) \in \mathbb{R}^n, \quad i \in \mathcal{V}, \tag{2}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^m$ are the state and the control input of the ith agent respectively; and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are defined as the system matrix and control input matrix. We assume that the agents are communicating over simple directed graphs, i.e., graphs