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Brief paper

Output feedback Continuous Twisting Algorithm☆

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ABSTRACT

Two output feedback controllers based on the Continuous Twisting Algorithm are provided. In those controllers, the state observers are based on the first and the second order Robust Exact Differentiators. The stability of the closed loops is proven through input-to-state stability properties. In the case of the second order differentiator, the conservation of homogeneity allows the output feedback scheme to preserve the robustness and accuracy properties of the state feedback Continuous Twisting Algorithm. In the same case, a smooth homogeneous Lyapunov function is constructed for the closed loop. A separation principle in the design of the controller and the observers is established. A qualitative analysis of the performance of the controllers in the presence of noise in the measurement is carried out. One of the schemes is used for output feedback control of a class of nonlinear systems.

1. Introduction

Sliding Mode Control is a useful technique to design controllers and observers for uncertain systems, providing robustness, and even insensibility, against some sort of disturbances (Levant, 2003; Utkin, Guldner, & Shi, 2009). Consider, for example, the disturbed double integrator

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = u + \delta\,, \quad y = x_1 + \nu\,, \tag{1}$$

where $x = [x_1, x_2]^\top \in \mathbb{R}^2$ is the state, y is the available output, $u \in \mathbb{R}$ is the control input, $\delta(t) \in \mathbb{R}$ is a Lipschitz disturbance, and $\nu(t) \in \mathbb{R}$ is a bounded noise signal. For $\nu(t) \equiv 0$, first-order sliding mode controllers (Utkin et al., 2009) are able to stabilize exponentially the origin of (1) by confining in finite-time the system dynamics in a desired sliding surface. However, this is done by using a discontinuous control signal causing the (generally undesirable) phenomenon of *chattering*, and the disturbance δ must be bounded. To substitute the discontinuous control signal with a continuous one, Super-Twisting controller (Levant, 1993, 1998) was suggested to stabilize exponentially the origin of (1) by reducing in finite-time the system dynamics in a desired sliding surface for the case when the disturbance δ is Lipschitz. Continuous controllers, such as those in Bernuau, Perruquetti, Efimov, and Moulay (2015) and Bhat and Bernstein (1998), achieve finite-time stability but are not able to reject Lipschitz disturbances δ.

Higher-order sliding mode (HOSM) controllers can be used to stabilize in finite-time the origin of (1). For example, Twisting, Terminal, Sub-Optimal and Quasi-Continuous controllers (Bartolini, Ferrara, & Usai, 1997; Levant, 1993, 2005b; Man, Paplinski, & Wu, 1994) ensure finite-time stability of the system's origin when δ is bounded, however, they also produce the discontinuous control action. The continuous HOSM controllers (Edwards & Shtessel, 2016; Kamal, Moreno, Chalanga, Bandyopadhyay, & Fridman, 2016; Laghrouche, Harmouche, & Chitour, 2017; Torres-González, Sanchez, Fridman, & Moreno, 2017), achieve finite-time stability of $x = 0$ despite Lipschitz disturbances δ. Remarkably, this is attained by means of a continuous control signal. These features make continuous HOSM very appealing, however, the performance of such controllers should be analysed considering additional issues present in real applications. For example, under discretization, finite-time convergence to the origin cannot be obtained. Nonetheless, these controllers exhibit, under discretization, an accuracy in steady state of order three (Kamal et al., 2016; Laghrouche et al., 2017; Torres-González et al., 2017). This is a guarantee of large steady state error reduction when the discretization step is reduced (Levant, 1993, 2005a). Other important issues is that the measurement of x_2 can be unavailable and the measurement of x_1

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can be noisy. In Chalanga, Kamal, Fridman, Bandyopadhyay, and Moreno (2016) the problem of output feedback for (1) is studied using the Super-Twisting controller by designing a sliding variable. It is shown that the Super-Twisting based observer cannot be applied for such a control strategy and a second order Robust Exact Differentiator (RED) (Levant, 2003) must be used to realize the controller's properties. Therefore, the problem of output feedback control of the continuous HOSM deserves special attention. To solve such a problem, the following options can be considered:

(a) The uniform differentiator proposed in Angulo, Moreno, and Fridman (2013) and Cruz-Zavala, Moreno, and Fridman (2011) is able to compute the derivative of x_1 in a fixed time (this time does not depend on the initial error). Then, it is possible to maintain the control off and turn it on after the uniform differentiator has converged. The disadvantage with this strategy is that the convergence time for the differentiator is usually overestimated, producing large time transients.

(b) The proposal in Angulo, Fridman, and Levant (2012) consists in using the RED to estimate the derivatives of the output. In this strategy, the controller must be maintained off until an on-line algorithm detects that the differentiator has converged.

An additional problem is that, for the noise analysis, the results of ISS robustness for homogeneous systems are not useful (in general) for the case of controllers involving discontinuities, see e.g. Bernuau, Efimov, Perruquetti, and Polyakov (2014) and Perruquetti (2018).

This paper devoted to the output feedback control for Continuous Twisting Algorithm (CTA) (Torres-González et al., 2017) that is able, in absence of noise, to stabilize in finite-time the origin of (1), compensating exactly Lipschitz disturbances δ ensuring accuracy of order three with respect to the output during discretization (Torres-González et al., 2017). The contributions can be summarized as follows.

(1) The robustness properties of the CTA considering noise in the states are studied through a Lyapunov function (LF).

(2) For (1), two output feedback schemes based on the first and second order REDs are considered. The effect caused by a noisy output y is investigated, and a separation principle based on input-to-state stability (ISS) properties of CTA and RED is provided.

(3) For the case of the second order RED a LF for the whole closed loop (CL) is designed. It is verified that in this case the conservation of homogeneity allows the output feedback scheme to preserve the robustness and accuracy properties of the state feedback CTA.

(4) It is shown how the scheme CTA–second order RED can be applied for output feedback control for the class of second order nonlinear systems that can be written as

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = f(x) + u + \delta\,, \quad y = x_1\,, \tag{2}$$

where x, u, δ, y are as above, and $f:\mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz function, for which we know a model $\bar{f}$ and a Lipschitz constant $l_f \in \mathbb{R}_{\geq 0}$.

(5) Numerical simulations are performed illustrating that the usage of second order RED allows to realize the third order accuracy predicted for CTA for output based CTA.

Paper organization: In Section 2 we recall the state feedback CTA and two RED observers. In this section we also provide the result on the robustness of the CTA in presence of noise. The output feedback controllers are stated in Section 3. The application to nonlinear systems with drift term is presented in Section 4. A numerical example is shown in Section 5. Some concluding remarks are given in Section 6.

Notation: $\mathbb{R}$ is the set of real numbers, and $\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$ (analogously for $\mathbb{R}_{\geq 0}$). For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm. $\mathbb{L}_n : \mathbb{R} \to \mathbb{R}^{n\times n}$ denotes the matrix $\mathbb{L}_n = \mathrm{diag}(L, \dots, L)$ for some $L \in \mathbb{R}_{\geq 0}$. For $x \in \mathbb{R}$ and $q \in \mathbb{R}_{\geq 0}$, $\lceil x \rfloor^q = \mathrm{sign}(x)|x|^q$.

2. The state feedback CTA and two observers

The state feedback CTA, given by

$$\begin{aligned} u &= -L^{\frac{2}{3}} k_1 \lceil x_1 \rfloor^{\frac{1}{3}} - L^{\frac{1}{2}} k_2 \lceil x_2 \rfloor^{\frac{1}{2}} + \eta\,, \\ \dot{\eta} &= -L k_3 \lceil x_1 \rfloor^0 - L k_4 \lceil x_2 \rfloor^0\,, \end{aligned} \tag{3}$$

is able to drive the states of (1) to zero in finite-time rejecting disturbances δ with bounded derivative (Torres-González et al., 2017). Although the second equation in (3) is discontinuous, this is integrated through η, allowing the control signal to be continuous. By defining the virtual state

$$x_3 = \eta + \delta(t)\,, \tag{4}$$

the CL (1), (3) is given by

$$\begin{aligned} \dot{x}_1 &= x_2\,, \\ \dot{x}_2 &= -L^{\frac{2}{3}} k_1 \lceil x_1 \rfloor^{\frac{1}{3}} - L^{\frac{1}{2}} k_2 \lceil x_2 \rfloor^{\frac{1}{2}} + x_3\,, \\ \dot{x}_3 &= -L k_3 \lceil x_1 \rfloor^0 - L k_4 \lceil x_2 \rfloor^0 + \dot{\delta}(t)\,. \end{aligned} \tag{5}$$

The third equation of (5) can be associated with the differential inclusion (DI) $\dot{x}_3 \in -k_3\lceil x_1 \rfloor^0 - k_4 \lceil x_2 \rfloor^0 + [-\Delta, \Delta]$ where $\lceil 0 \rfloor^0 = [-1, 1] \subset \mathbb{R}$. Hence, the solutions of (5) are understood in the sense of Filippov (1988). The DI associated to (5) is $\mathbf{r}$-homogeneous of degree $\kappa = -1$ with weights $\mathbf{r} = [3, 2, 1]^\top$ (see Appendix A for homogeneity definitions). We recall the following theorem from Torres-González et al. (2017).

Theorem 1 (*Torres-González et al., 2017*)**.** *Consider the* $\mathbf{r}$*-homogeneous function* $V_1 : \mathbb{R}^3 \to \mathbb{R}$*, of degree* $m = 5$*, given by*

$$\begin{aligned} V_1(x) = {} & \alpha_1 |x_1|^{\frac{5}{3}} + \alpha_2 x_1 x_2 + \alpha_3 |x_2|^{\frac{5}{2}} - \\ & \alpha_4 x_1 \lceil x_3 \rfloor^2 - \alpha_5 x_2 x_3^3 + \alpha_6 |x_3|^5\,. \end{aligned} \tag{6}$$

If $|\dot{\delta}(t)| \leq L = 1$*, then there exist gains* $k = [k_1, \dots, k_4]^\top$ *and coefficients* $\alpha = [\alpha_1, \dots, \alpha_6]^\top \in \mathbb{R}^6$ *such that the origin of (1) is finite-time stable.*[2] *in CL with (3), and (6) is a LF for (5). Moreover, for such* k *and* α*, if* $|\dot{\delta}(t)| \leq \Delta$ *and* $L \geq \Delta$*, then the origin of (1) is finite-time stable in CL with (3), and* $\bar{V}_1(x) = V_1(\mathbb{L}_3^{-1} x)$ *is a LF for (5).*

Now, we consider the CTA with noise inputs, i.e.

$$\begin{aligned} \dot{x}_1 &= x_2\,, \\ \dot{x}_2 &= -L^{\frac{2}{3}} k_1 \lceil x_1 + \nu_1 \rfloor^{\frac{1}{3}} - L^{\frac{1}{2}} k_2 \lceil x_2 + \nu_2 \rfloor^{\frac{1}{2}} + x_3\,, \\ \dot{x}_3 &= -L k_3 \lceil x_1 + \nu_1 \rfloor^0 - L k_4 \lceil x_2 + \nu_2 \rfloor^0 + \dot{\delta}(t)\,, \end{aligned} \tag{7}$$

where $\nu_1 = \nu_1(t) \in \mathbb{R}$ and $\nu_2 = \nu_2(t) \in \mathbb{R}$ are such that $|\nu_1(t)| \leq N_1$ and $|\nu_2(t)| \leq N_2$ for all $t \geq 0$ for some $N_1, N_2 \in \mathbb{R}_{\geq 0}$.

Theorem 2. *Consider (7) and suppose that for* $\nu_i(t) \equiv 0$ *Theorem 1 holds. Then* $\bar{V}_1$ *is an ISS-LF for (7). Moreover, for any finite* $N_1, N_2 \in \mathbb{R}_{\geq 0}$*, there exist constants* $\theta_i \in \mathbb{R}_{>0}$*,* $i = 1, 2, 3$*, such that the bounds* $|x_i(t)| \leq \theta_i \left(N_1^{\frac{r_i}{3}} + N_2^{\frac{r_i}{2}} \right)$*,* $\mathbf{r} = [3, 2, 1]^\top$*, are established in finite-time.*

Observe that the existent results on ISS robustness for homogeneous systems (see e.g. Bernuau et al., 2014, Perruquetti, 2018) cannot be used to prove the particular case of Theorem 2. The proof of Theorem 2 is in Appendix B.

From (1) we have that $x_2 = \dot{y}$, therefore, it is clear that a first order differentiator is sufficient to obtain the estimation of the second state. Hence, we consider the following first order RED observer (Davila, Fridman, & Levant, 2005; Levant, 1998),

$$\dot{\hat{x}}_1 = -\bar{L}^{\frac{1}{2}} l_1 \lceil \hat{x}_1 - y \rfloor^{\frac{1}{2}} + \hat{x}_2,\ \ \dot{\hat{x}}_2 = -\bar{L} l_2 \lceil \hat{x}_1 - y \rfloor^0 + u\,. \tag{8}$$

[2] For the definition of finite-time stability see e.g. Orlov (2003).