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Brief paper

Minimal realizations of nonlinear systems ☆

Ülle Kotta [a], Claude H. Moog [b], Maris Tõnso [a,*]

[a] Department of Software Science, Tallinn University of Technology, Estonia
[b] LS2N UMR CNRS 6004, 1, rue de la Noë, BP 92101, 44321 Nantes, France

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ABSTRACT

The nonlinear realization theory is recasted for time-varying single-input single-output nonlinear systems. The concept of realization has been extended to cover also the realizations with order greater than the order of input–output equation. The minimal realization problem is studied. The state realization is said to be minimal if it is either accessible and observable or its state dimension is minimal. In the linear case the two definitions are equivalent, but not for nonlinear time-invariant systems. It is shown that the two definitions remain equivalent for nonlinear systems under certain technical assumptions. Two alternative methods are presented for finding the minimal realization.

1. Introduction

There exist numerous papers where the realization of nonlinear time-invariant systems is studied, see for instance Belikov, Kotta, and Tõnso (2014), Conte, Moog, and Perdon (2007), Delaleau and Respondek (1995) and van der Schaft (1987), but to the best knowledge of the authors there is no contribution to the realization problem of nonlinear time-varying systems. We follow the algebraic approach of differential one-forms (Conte et al., 2007), combined with the theory of non-commutative polynomial rings (Belikov et al., 2014; Halás, 2008; Zhang, Moog, & Xia, 2010; Zheng, Willems, & Zhang, 2001), adapted from time-invariant to time-varying single-input single-output (SISO) case. In the present paper (i) a new definition of (transfer) equivalence and realization are given, (ii) realizability conditions in Proposition 3 have been generalized from time-invariant to time-varying systems and more importantly, extended also for the case when the dimension of realization is greater than the order of input–output (i/o) equation. Recall that in the literature two definitions of minimality of the state space realization are used. First, one may require minimality of the state dimension. Second, the realization is said to be minimal when it is both observable and accessible (controllable), see for instance Kailath (1980), 363. Though in the linear time-invariant case these two definitions are equivalent, this is no longer true in the class of nonlinear time-invariant systems, as shown via examples in Zhang et al. (2010). The latter points to the inconsistency of linear and nonlinear theories. We will show that these two definitions remain equivalent under certain technical assumptions.

In general, the direct application of the realization algorithm does not necessarily provide a realization with minimal state dimension. To find the minimal realization of time-varying nonlinear system, two alternative approaches are considered in the paper. The *first* approach is based on the fact that if one starts from the irreducible i/o equation, then the realization will be accessible.[1] Reduction theory of nonlinear systems is based on the notion of irreducible variable φ, i.e. a variable, satisfying certain autonomous differential equation $F(\varphi, \varphi^{(1)}, \dots, \varphi^{(\mu)}) = 0$, see for instance Conte et al. (2007) and Zhang et al. (2010); Zheng et al. (2001). The irreducible equation is formed from the assumption $F(0, \dots, 0) = 0$, taking $\varphi = 0$.[2] The *second* method starts from a non-minimal realization, followed by the decomposition of the latter into non-accessible and accessible subsystems. Then the equations of the accessible subsystem may be taken as the minimal observable realization of the given i/o equation after substituting into it the solution of the non-accessible part.

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* Corresponding author.
E-mail addresses: kotta@cc.ioc.ee (Ü. Kotta), Claude.Moog@irccyn.ec-nantes.fr (C.H. Moog), maris@cc.ioc.ee (M. Tõnso).

1 The realization algorithms always yield an observable realization.

2 This means that zero is a solution of the autonomous differential equation. The assumption is reasonable for majority of nonlinear systems and for all linear systems.

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The paper is organized as follows. Section 2 recalls the basics of the algebraic approach. In Section 3 the notion of equivalent one-forms is introduced. Section 4 deals with the reduction problem. In Section 5 the general realization of arbitrary dimension is considered and in Section 6 the minimal realization is discussed. Finally, Section 7 draws conclusions.

2. Preliminaries

In this paper two types of SISO nonlinear time-varying equations are considered. First, the i/o equation in the form

$$y^{(n)}(t) = \phi(t,\ y(t), \dots, y^{(n-1)}(t), u(t), \dots, u^{(r)}(t)), \tag{1}$$

and second, the state equations

$$\dot{x}(t) = f(t, x(t), u(t)),\ \ y(t) = h(t, x(t)), \tag{2}$$

where $u(t) \in \mathbb{R}$ is input, $y(t) \in \mathbb{R}$ is output and $x(t) \in \mathbb{R}^n$ is state variable. For the sake of compactness the argument t will be omitted from now on. The special case, where input u is missing in systems (1) or (2), is not treated in this paper. Sometimes also the i/o equation in implicit form are considered

$$\psi(t, y, \dots, y^{(n)}, u, \dots, u^{(r)}) = 0, \tag{3}$$

where $\psi(\cdot) = y^{(n)} - \phi(\cdot)$. Additionally, we assume that $\psi(\cdot)$ and $f(t, x, u)$ (in the expanded form) do not include the terms, depending only on t. This requirement is consistent with linear theory which considers $\dot{x} = A(t)x + B(t)u$, and not $\dot{x} = A(t)x + B(t)u + C(t)$. Below we briefly recall the approach of differential 1-forms from Conte et al. (2007), extending it to the time-varying case, i.e. for the case when the system equations depend explicitly on time t. Formally, this means that the ground field $k = \mathbb{R}(t)$ is a field of meromorphic functions of t and not just $\mathbb{R}$ as in the case of time-invariant systems. The approach of 1-forms is based on the idea of working with differentials of nonlinear system equations rather than with equations themselves. This allows to linearize the intermediate computations.

Let $\mathcal{A}_\infty$ be the ring of analytic functions in a finite number of variables from the set $\{t,\ y^{(\ell)}, \ell \geqslant 0,\ u^{(k)}, k \geqslant 0\}$. The ring $\mathcal{A}_\infty$ is an integral domain. Let $\mathcal{K}_\infty$ be the field of fractions of the ring $\mathcal{A}_\infty$. The elements of $\mathcal{K}_\infty$ are the meromorphic functions. Let $\mathrm{d}/\mathrm{d}t : \mathcal{K}_\infty \to \mathcal{K}_\infty$ be the time-derivation operator. For the sake of compactness we write $\mathrm{d}/\mathrm{d}t(a) := \dot{a}$, $(\mathrm{d}/\mathrm{d}t)^2(a) := \ddot{a}$ and $(\mathrm{d}/\mathrm{d}t)^n(a) := a^{(n)}$ for $n > 2$, $a \in \mathcal{K}_\infty$. Then the pair $(\mathcal{K}_\infty, \mathrm{d}/\mathrm{d}t)$ is differential field (Kolchin, 1973). Over the field $\mathcal{K}_\infty$ a differential vector space $\mathcal{E}_\infty := \mathrm{sp}_{\mathcal{K}_\infty}\{\mathrm{d}\zeta \mid \zeta \in \mathcal{K}_\infty\}$ is defined, where sp denotes linear span. Consider a 1-form $\omega \in \mathcal{E}_\infty$ such that $\omega = \sum_i \alpha_i \mathrm{d}\zeta_i$, $\alpha_i, \zeta_i \in \mathcal{K}_\infty$. Its derivative $\dot{\omega}$ is defined by $\dot{\omega} = \sum_i(\dot{\alpha}_i \mathrm{d}\zeta_i + \alpha_i \mathrm{d}\dot{\zeta}_i)$. The same notations are used for derivative operators in $\mathcal{K}_\infty$ and $\mathcal{E}_\infty$. The space $\mathcal{E}_\infty$ is closed under derivative operator. One says that $\omega \in \mathcal{E}_\infty$ is an *exact* 1-form if $\omega = \mathrm{d}\alpha$ for some $\alpha \in \mathcal{K}_\infty$. A 1-form ν for which $\mathrm{d}\nu = 0$ is said to be *closed* (locally exact). A subspace $\mathcal{V}$ is said to be closed or *completely integrable*, if it admits locally an exact basis $\mathcal{V} = \mathrm{sp}_{\mathcal{K}_\infty}\{\mathrm{d}\zeta_1, \dots, \mathrm{d}\zeta_r\}$ (Choquet-Bruhat, DeWitt-Morette, & Dillard-Bleichi, 1982). Integrability of $\mathcal{V} = \mathrm{sp}_{\mathcal{K}_\infty}\{\omega_1, \dots, \omega_r\}$ can be checked by *Frobenius theorem*: $\mathcal{V}$ is completely integrable if and only if $\mathrm{d}\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_r = 0$ for $i = 1, \dots, r$. Here d is exterior differential operator and $\wedge$ denotes the wedge product, see Choquet-Bruhat et al. (1982).

Next, the algebraic approach of 1-forms is supplemented by the theory of non-commutative polynomial ring. Polynomials allow to represent the 1-forms as well as the operations with them in a compact form; such tools have been used to address many problems for nonlinear time-invariant systems (Belikov, Kotta, & Tõnso, 2015; Halás, 2008; Zheng et al., 2001). The field $\mathcal{K}_\infty$ and the operator $\mathrm{d}/\mathrm{d}t$ induce a non-commutative ring of left differential polynomials $\mathcal{K}_\infty[s]$. A polynomial $p \in \mathcal{K}_\infty[s]$ can be uniquely written as $p = p_\kappa s^\kappa + p_{\kappa-1}s^{\kappa-1} + \cdots + p_1 s + p_0$, where s is a formal variable and $p_i \in \mathcal{K}_\infty$ for $i = 0, \dots \kappa$. Polynomial $p \neq 0$ if and only if at least one of the functions p_i is non-zero. If $p_\kappa \not\equiv 0$, then the integer κ is called the *degree* of p and denoted by $\deg(p)$. We set additionally $\deg(0) = -\infty$. The addition of the polynomials is defined in the standard way. However, for $a \in \mathcal{K}_\infty \subset \mathcal{K}_\infty[s]$ the multiplication is defined by the commutation rule $s \cdot a := a s + \dot{a}$. In $\mathcal{K}_\infty[s]$ the following *left Ore condition* holds: For all non-zero $a, b \in \mathcal{K}_\infty[s]$ there exist non-zero $\widetilde{a}, \widetilde{b} \in \mathcal{K}_\infty[s]$ such that $\widetilde{a}a = \widetilde{b}b$. From the ring $\mathcal{K}_\infty[s]$ one can construct a non-commutative *field of fractions*. Define a set $\mathcal{V} := \mathcal{K}_\infty[s] \setminus \{0\}$. Consider the set of left fractions denoted by $\mathcal{K}_\infty(s) = \mathcal{V}^{-1}\mathcal{K}_\infty[s]$. Elements of $\mathcal{K}_\infty(s)$ are left fractions in the form $b^{-1}a$, where $a \in \mathcal{K}_\infty[s]$, $b \in \mathcal{V}$. Since the ring $\mathcal{K}_\infty[s]$ is an integral domain, thus $\mathcal{K}_\infty(s)$ is a field.

A left differential polynomial $a \in \mathcal{K}_\infty[s]$ may be interpreted as an operator $a(s) : \mathcal{E}_\infty \to \mathcal{E}_\infty$. Define for $\mathrm{d}y, \mathrm{d}u, \mathrm{d}t \in \mathcal{E}_\infty$

$$\begin{aligned} &s\mathrm{d}y := (\mathrm{d}/\mathrm{d}t)\mathrm{d}y = \mathrm{d}\dot{y}, \quad s\mathrm{d}u := (\mathrm{d}/\mathrm{d}t)\mathrm{d}u = \mathrm{d}\dot{u}, \\ &s\mathrm{d}t := (\mathrm{d}/\mathrm{d}t)\mathrm{d}t = 0. \end{aligned} \tag{4}$$

It is natural to extend (4) for $a = \sum_{i=0}^k a_i s^i$ as $a(s)(\alpha \mathrm{d}\zeta) := \sum_{i=0}^k a_i(s^i \cdot \alpha)\mathrm{d}\zeta$ with $a_i, \alpha \in \mathcal{K}_\infty$ and $\mathrm{d}\zeta \in \{\mathrm{d}y, \mathrm{d}u, \mathrm{d}t\}$. Using (4) every 1-form $\omega = \sum_{\alpha=0}^k a_\alpha \mathrm{d}y^{(\alpha)} + \sum_{\beta=0}^\ell b_\beta \mathrm{d}u^{(\beta)} + c_0\mathrm{d}t \in \mathcal{E}_\infty$, where $a_\alpha, b_\beta, c_0 \in \mathcal{K}_\infty$, may be expressed in terms of the left differential polynomials as $\omega = \sum_{\alpha=0}^k a_\alpha s^\alpha \mathrm{d}y + \sum_{\beta=0}^\ell b_\beta s^\beta \mathrm{d}u + c_0\mathrm{d}t := a(s)\mathrm{d}y + b(s)\mathrm{d}u + c\mathrm{d}t$, where $a, b \in \mathcal{K}_\infty[s]$ and $c = c_0 \in \mathcal{K}_\infty$. It is easy to see that $s\,\omega = \dot{\omega}$, for $\omega \in \mathcal{E}_\infty$.

3. Equivalence of 1-forms

In Section 4 we associate with each nonlinear system its 1-form and on the set of 1-forms $\mathcal{E}_\infty$ the equivalence relation is defined. In linear case the equivalence relation reduces to the equality of transfer functions.

Definition 1. The 1-forms $\omega_1, \omega_2 \in \mathcal{E}_\infty$ are called *equivalent*, denoted $\omega_1 \cong \omega_2$, if there exist non-zero polynomials $\lambda, \mu \in \mathcal{K}_\infty[s]$ such that

$$\lambda(s)\omega_1 = \mu(s)\omega_2. \tag{5}$$

Proposition 1. *Relation (5) defines an equivalence relation on $\mathcal{E}_\infty$.*

Proof. Symmetry and reflexivity are obvious. To show transitivity we assume that $\omega \cong \widetilde{\omega}$ and $\widetilde{\omega} \cong \hat{\omega}$. Due to (5) we may write

$$\alpha(s)\omega = \beta(s)\widetilde{\omega},\ \ \gamma(s)\widetilde{\omega} = \delta(s)\hat{\omega}, \tag{6}$$

By the left Ore condition one can find, for arbitrary non-zero polynomials $\beta, \gamma \in \mathcal{K}_\infty[s]$, two non-zero $\widetilde{\beta}, \widetilde{\gamma} \in \mathcal{K}_\infty[s]$ such that $\widetilde{\beta}\beta = \widetilde{\gamma}\gamma$. Multiplying the relations (6), respectively, by $\widetilde{\beta}$ and $\widetilde{\gamma}$ from left, we obtain $\widetilde{\beta}(s)\alpha(s)\omega = \widetilde{\beta}(s)\beta(s)\widetilde{\omega}$, $\widetilde{\gamma}(s)\gamma(s)\widetilde{\omega} = \widetilde{\gamma}(s)\delta(s)\hat{\omega}$. Adding these equalities and regarding that $\widetilde{\beta}\beta = \widetilde{\gamma}\gamma$ yields $\widetilde{\beta}(s)\alpha(s)\omega = \widetilde{\gamma}(s)\delta(s)\hat{\omega}$, thus $\omega \cong \hat{\omega}$. □

The equivalence relation divides 1-forms in $\mathcal{E}_\infty$ into equivalence classes.

Definition 2. A differential form $\omega \in \mathcal{E}$ is called *irreducible*, if $\omega = \gamma(s)\pi$, where $\gamma \in \mathcal{K}_\infty[s]$, $\pi \in \mathcal{E}_\infty$, and $\pi \neq 0$ yields $\deg(\gamma) = 0$.

Definition 3. Given $\omega \in \mathcal{E}_\infty$, the form π_ω is an irreducible form of ω whenever $\pi_\omega \cong \omega$ and π_ω is irreducible.

Algorithm 1 finds $\pi_\omega = \tilde{a}(s)\mathrm{d}y + \tilde{b}(s)\mathrm{d}u + \tilde{c}\mathrm{d}t$ for the given 1-form $\omega = a(s)\mathrm{d}y + b(s)\mathrm{d}u + c\mathrm{d}t$.