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Asymptotic stability for a class of boundary control systems with non-linear damping ⋆

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Abstract: The asymptotic stability of boundary controlled port-Hamiltonian systems defined on a 1D spatial domain interconnected to a class of non-linear boundary damping is addressed. It is shown that if the port-Hamiltonian system is approximately observable, then any boundary damping which behaves linear for small velocities asymptotically stabilizes the system.

Keywords: Boundary control systems, infinite-dimensional port Hamiltonian systems, asymptotic stability, non-linear control.

1. INTRODUCTION

Many physical distributed parameter systems can be controlled through their boundaries. This is for instance the case for transmission lines, flexible beams and plates, tubular and nuclear fusion reactors and so on. This class of systems is called Boundary Controlled Systems (BCS). In the linear case the control design for such system can be tackled using the semigroup theory and the associated abstract formulation based on unbounded input/output mappings (Curtain and Zwart, 1995; Staffans, 2005). When asymptotic or exponential stability by non-linear control is concerned, the main difficulty remains in finding the appropriate Lyapunov function candidate to prove the stability. It is usually done on a case by case basis using physical considerations depending on the application field.

In the last decade, an alternative approach has been developed in order to deal with a large class of physical systems. This approach is based on the extension of the Hamiltonian formulation to open distributed parameter systems (van der Schaft and Maschke, 2002). In the 1D linear case it gave rise to the definition of boundary controlled port Hamiltonian systems (Le Gorrec et al., 2004) and allowed to parametrize all the possible boundary conditions that define a boundary control system (Le Gorrec et al., 2005) by using simple matrix conditions. Many variations around these primary works can be found in (Villegas, 2007) and in (Jacob and Zwart, 2012). Well posedness and stability have been investigated in open-loop and in the case of static boundary feedback control in (Zwart et al., 2010) and (Villegas et al., 2005; Villegas et al., 2009), respectively, and in the case of dynamic linear control in (Ramirez et al., 2014; Augner and Jacob, 2014).

This paper is restricted to the analysis of the asymptotic stability of a port-Hamiltonian system connected to a non-linear damper. It is show that asymptotic stability can be proved whenever the port-Hamiltonian system is approximately observable. In the next section we introduce our class of port-Hamiltonian systems and our class of dampers. In Section 3 we formulate and prove our main theorem.

2. PORT-HAMILTONIAN SYSTEMS

The systems under study are described by the following 1D partial differential equation (PDE):

$$\frac{\partial x}{\partial t} = P_1 \frac{\partial}{\partial \zeta} \left(\mathcal{H}(\zeta)x(t,\zeta)\right) + P_0 \mathcal{H}(\zeta)x(t,\zeta), \tag{1}$$

$\zeta \in (a,b)$, where $P_1 \in M_n(\mathbb{R})$ [1] is a non-singular symmetric matrix, $P_0 = -P_0^\top \in M_n(\mathbb{R})$, and x takes values in $\mathbb{R}^n$. Furthermore, $\mathcal{H}(\cdot) \in L_\infty((a,b); M_n(\mathbb{R}))$ is a bounded and measurable, matrix-valued function satisfying for almost all $\zeta \in (a,b)$, $\mathcal{H}(\zeta) = \mathcal{H}(\zeta)^\top$ and $\mathcal{H}(\zeta) > mI$, with m independent from ζ.

For simplicity $\mathcal{H}(\zeta)x(t,\zeta)$ will be denoted by $(\mathcal{H}x)(t,\zeta)$. For the above pde we assume that some boundary conditions are homogeneous, whereas others are controlled. Thus there are matrices of appropriate sizes such that

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[1] $M_n(\mathbb{R})$ denote the space of real $n \times n$ matrices

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$$u(t) = W_{B,1} \begin{bmatrix} (\mathcal{H}x)(t,b) \\ (\mathcal{H}x)(t,a) \end{bmatrix} \tag{2}$$

and

$$0 = W_{B,2} \begin{bmatrix} (\mathcal{H}x)(t,b) \\ (\mathcal{H}x)(t,a) \end{bmatrix}. \tag{3}$$

Furthermore, there is a boundary output given by

$$y(t) = W_C \begin{bmatrix} (\mathcal{H}x)(t,b) \\ (\mathcal{H}x)(t,a) \end{bmatrix}. \tag{4}$$

To study the existence and uniqueness of solution to the above controlled pde, we follow the semigroup theory, see also Le Gorrec et al. (2005); Jacob and Zwart (2012). Therefor we define the state space X as $X = L_2((a,b);\mathbb{R}^n)$ with inner product $\langle x_1, x_2 \rangle_{\mathcal{H}} = \langle x_1, \mathcal{H}x_2 \rangle$ and norm $\|x\|^2_{\mathcal{H}} = \langle x, x \rangle_{\mathcal{H}}$. Note that the norm on X and the L_2 norm are equivalent. Hence X is a Hilbert space. The reason for selecting this space is that $\|\cdot\|^2_{\mathcal{H}}$ is related to the energy function of the system, i.e., the total energy of the system equals $E(t) = \frac{1}{2}\|x\|^2_{\mathcal{H}}$. The Sobolev space of order k is denoted by $H^k((a,b),\mathbb{R}^n)$.

Associated to the (homogeneous) pde we define the operator $Ax = P_1(\partial/\partial\zeta)(\mathcal{H}x) + P_0\mathcal{H}x$ with domain

$$D(A) = \left\{ \mathcal{H}x \in H^1((a,b);\mathbb{R}^n) \;\middle|\; \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix} \in \ker W_B \right\}$$

where $W_B = \left[\begin{smallmatrix} W_{B,1} \\ W_{B,2} \end{smallmatrix}\right]$. For the rest of the paper we make the following hypothesis

Hypothesis 1. For the operator A and the pde (1)–(4) the following hold:

(1) The matrix W_B is an $n \times 2n$ matrix of full rank;
(2) For $x_0 \in D(A)$ we have $\langle Ax_0, x_0 \rangle_{\mathcal{H}} \leq 0$.
(3) The number of inputs and outputs are the same, k, and for classical solutions of (1)–(4) there holds $\dot{E}(t) = u(t)^\top y(t)$.

We remark that for the system at hand, condition (2) could be replaced by $W_B \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] W_B^\top \geq 0$. Furthermore, from hypothesis (1) and (2) it follows that the system (1)–(4) is a boundary control system (see Le Gorrec et al. (2005); Jacob and Zwart (2012); Jacob et al. (2015)), and so for $u \in C^2([0,\infty);\mathbb{R}^k)$, $\mathcal{H}x(0) \in H^1((a,b);\mathbb{R}^n)$, satisfying (2) and (3) (for $t = 0$), there exists a unique classical solution to (1)–(4). Thus for these dense sets of initial conditions and inputs hypothesis (3) makes sense.

Although the above formulation can be used to describe many models on different physical domain, we regard the above port-Hamiltonian system to describe a mechanical system in which u represents (generalised) boundary velocities, and y are the (generalised) boundary forces, the controller is regarded as a generalised mass-damper system. The associated momenta and velocities are denoted by p and v, respectively, and they are related via the mass matrix M, i.e., $p = Mv$. Using Newton's second law, we find

$$\dot{p} = f_{pH} + f_d, \tag{5}$$

where f_{pH} is the force felt from the port-Hamiltonian system, and f_d is the reactive damping force. Based on the interconnecting as discussed above we have the following interconnection relation between the two systems

$$f_{pH} = -y, \qquad v = u. \tag{6}$$

The state space for the closed loop system equals the direct sum of the separate state spaces, i.e. $X_{\text{ext}} = X \oplus \mathbb{R}^k$. The norm is given by

$$\left\| \begin{bmatrix} x \\ v \end{bmatrix} \right\|^2 = \|x\|^2_{\mathcal{H}} + v^\top M v. \tag{7}$$

Hence we have that the norm equals twice the total energy.

The closed loop system now becomes

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} P_1 \dfrac{\partial}{\partial\zeta}(\mathcal{H}x) + P_0\mathcal{H}x \\ -M^{-1}W_C \begin{bmatrix} (\mathcal{H}x)(t,b) \\ (\mathcal{H}x)(t,a) \end{bmatrix} \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}f_d \end{bmatrix}. \tag{8}$$

Furthermore, (3) holds together with

$$v(t) = W_{B,1} \begin{bmatrix} (\mathcal{H}x)(t,b) \\ (\mathcal{H}x)(t,a) \end{bmatrix}. \tag{9}$$

We see that we can write the above as the abstract system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = A_{\text{ext}} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} f_d(t) \tag{10}$$

with A_{ext} given by the corresponding expression in (8) with domain

$$D(A_{\text{ext}}) = \Bigg\{ \mathcal{H}x \in H^1((a,b);\mathbb{R}^n), v \in \mathbb{R}^k \;\Bigg|\; v = W_{B,1} \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix},\ 0 = W_{B,2} \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix} \Bigg\}.$$

By using similar arguments as in Ramirez et al. (2014) it can be shown that A_{ext} with its domain generates a contraction semigroup on X_{ext}. Moreover, since H^1 is compactly embedded into L_2, we have that A_{ext} has a compact resolvent.

The following energy balance equation will be useful in the next section. Along classical solutions of (8) there holds

$$\begin{aligned} \dot{E}_{\text{tot}}(t) &= \dot{E}(t) + v(t)^\top M \dot{v}(t) \\ &= u(t)^\top y(t) + v(t)^\top M(-M^{-1}y(t) + M^{-1}f_d(t)) \\ &= v(t)^\top f_d(t). \end{aligned} \tag{11}$$

From this equality we see two things. Firstly, when we want to damp the system the damping force needs to be opposite to the velocity. Secondly, when we associate to the system (10) the output operator

$$C_{\text{ext}} = [0\ 1],$$

then $\dot{E}_{\text{tot}}(t)$ is again output times input and $C^*_{\text{ext}} = B_{\text{ext}} := \left[\begin{smallmatrix} 0 \\ M^{-1} \end{smallmatrix}\right]$. Note that the adjoint is calculated with respect to the inner product of X_{ext}.

3. ASYMPTOTIC STABILITY

As we have seen in the previous section, if we want that the energy decays, then we have to inject damping into the system. For the (generalised) damping force we assume the following.

Hypothesis 2. The damping force is a function of the velocity only, i.e., $f_d = -F(v)$. It is opposite the velocity, i.e.,

$$v^\top F(v) \geq 0, \qquad v \in \mathbb{R}^k.$$

Furthermore, the F is a locally Lipschitz continuous function, and there exist positive constants δ, α, γ such that $v^\top F(v) \geq \alpha\|v\|^2$ when $\|v\| < \delta$ and $v^\top F(v) \geq \gamma$ when $\|v\| \geq \delta$.