Automatica 95 (2018) 222–228

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Distributed sub-optimal resource allocation over weight-balanced graph via singular perturbation ☆

Shu Liang [a], Xianlin Zeng [b], Yiguang Hong [c,*]

[a] Key Laboratory of Knowledge Automation for Industrial Processes of Ministry of Education, School of Automation and Electrical Engineering, University of Science and Technology Beijing, Beijing 100083, China
[b] School of Automation, Beijing Institute of Technology, 100081, Beijing, China
[c] Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China

ARTICLE INFO

Article history:
Received 5 February 2017
Received in revised form 17 April 2018
Accepted 23 April 2018

Keywords:
Distributed optimization
Resource allocation
Sub-optimality
Weight-balanced graph
Singular perturbation

ABSTRACT

In this paper, we consider distributed optimization design for resource allocation problems over weight-balanced graphs. With the help of singular perturbation analysis, we propose a simple sub-optimal continuous-time optimization algorithm. Moreover, we prove the existence and uniqueness of the algorithm equilibrium, and then show the convergence with an exponential rate. Finally, we verify the sub-optimality of the algorithm, which can approach the optimal solution as an adjustable parameter tends to zero.

1. Introduction

Distributed optimization has attracted intense research attention in recent years, due to its theoretic significance and broad applications in various research fields, and much effort has also been done for distributed discrete-time or continuous-time algorithm design, referring to Gharesifard and Cortés (2014), Lou, Hong, and Wang (2016), Nedić and Ozdaglar (2009), Shi, Johansson, and Hong (2013), Yang, Liu, and Wang (2017) and Yuan, Ho, and Xu (2016).

Resource allocation is one of the most important optimization problems. It has been widely investigated in economic systems, communication networks, and power grids for optimization and identification (Cherukuri & Cortés, 2016; Guo, Mu, Wang, Yin, & Xu, 2017; Marden & Roughgarden, 2014; Zappone, Sanguinetti, Bacci, Jorswieck, & Debbah, 2016). Historic development of resource allocation algorithms can be found in Ibaraki and Katoh (1988). In this decade, more and more significant results were obtained. For example, Xiao and Boyd (2006) and Lakshmanan and De Farias (2008) developed a weighted gradient algorithm and an asynchronous gradient-descent algorithm for resource allocation problems, respectively, while Cherukuri and Cortés (2015) proposed a Laplacian-gradient dynamics for resource allocation problems. Also, Necoara and Nedelcu (2015) introduced a dual gradient algorithm that considers both equality and inequality constraints and achieves a linear rate under strong convexity, and Liang, Zeng, and Hong (2018) considered a resource allocation inequality constraint and developed a continuous-time design with the exact penalty. The algorithms (Cherukuri & Cortés, 2015; Xiao & Boyd, 2006) require the initial value to satisfy the resource constraints. Moreover, Cherukuri and Cortés (2016) and Yi, Hong, and Liu (2016) further developed distributed initialization-free algorithms to solve the resource allocation problem.

In distributed continuous-time design, with or without the resource allocation constraint, Gharesifard, Basar, and Dominguez-Garcia (2016) and Yi et al. (2016) dealt with undirected graph cases. Moreover, Cherukuri and Cortés (2016) and Gharesifard and Cortés (2014) considered balanced directed graphs, but their methods need additional computation for the maximum singular value of the Laplacians. Thus, the convergence of these distributed algorithms may be sensitive to the network topology, i.e., they can be divergent when the graph is slightly changed.

It may be very hard to get the exact solution for distributed optimization due to technical difficulties and complexity or cost, and sub-optimal algorithms may provide considerable benefits with simple feasible designs and even performance enhancement.

☆ This work was supported by National Key Research and Development Program of China (2016YFB0901902), NSFC (61733018, 61333001, 61573344, 61603378), and Fundamental Research Funds for the China Central Universities of USTB (No. FRF-TP-17-088A1). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Gurdal Arslan under the direction of Editor Ian R. Petersen.

* Corresponding author.
E-mail addresses: sliang@amss.ac.cn (S. Liang), xianlin.zeng@bit.edu.cn (X. Zeng), yghong@iss.ac.cn (Y. Hong).

https://doi.org/10.1016/j.automatica.2018.05.013
0005-1098/

However, there are quite few results about its sub-optimal algorithms and related analysis. For example, Nedić and Ozdaglar (2009) introduced an algorithm without exactly solving the considered problem, but with fast convergence rate. In fact, sub-optimal design may particularly deserve investigation over a large-scale network, to reduce the computational complexity or sensitivity to the network topology, rather than to seek high-cost exact optimal solutions.

Our motivation is to study a distributed sub-optimal algorithm design for the resource allocation optimization over a weight-balanced graph, which is of lower dimensions than existing ones to reduce computational burden and information exchanging. Moreover, its convergence is kept over any strongly connected and balanced graph because its design does not depend on any specific knowledge of the graph. To achieve this, we adopt a singular perturbation idea, which is quite powerful for (continuous-time) control (Kokotovic, Khalil, & O'reilly, 1999), in the distributed design. Note that the well-known high-gain technique and semi-global stabilization design are closely related to singular perturbation (Khalil, 2002). The contributions of this paper can be summarized as follows. (i) We propose a distributed initialization-free algorithm for a resource allocation problem over weight-balanced graphs, which achieves a sub-optimal solution. The design is simple, and we prove the existence and uniqueness of the algorithm equilibrium. (ii) We adopt a singular perturbation idea in our design, totally different from that given in Cherukuri and Cortés (2016) and Yi et al. (2016), and then show that the quasi-steady-state model of our algorithm is exactly the primal–dual optimization algorithm. Note that the original primal–dual algorithm may not be directly implementable in a fully distributed manner due to the coupled resource allocation constraint. (iii) We prove the exponential convergence of the proposed algorithm, and estimate the distance from the sub-optimal solution to the optimal one, which is bounded linearly by an adjustable parameter. Moreover, we verify that the sub-optimal solution always satisfies the resource allocation constraint and can be made arbitrarily close to the optimal point as the parameter tends to 0.

Notations: Let $\mathbb{R}^n$ be the n-dimensional real vector space and $\mathbb{B}$ be the unit ball. Let $\|\cdot\|$ be the Euclidean norm. $col\{x_1, \dots, x_N\}$ is the column vector stacked with $x_1, \dots, x_N$, and $1_n = col\{1, \dots, 1\} \in \mathbb{R}^n$. I_n is the identity matrix in $\mathbb{R}^{n\times n}$. $\otimes$ is the Kronecker's product for matrices. $\det(\cdot)$ is the determinant of a matrix.

2. Preliminaries and formulation

In this section, we introduce preliminary results and formulate the distributed problem.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *convex* if $f(\lambda z_1 + (1-\lambda)z_2) \le \lambda f(z_1) + (1-\lambda)f(z_2)$ for any $z_1, z_2 \in \mathbb{R}^n$ and $\lambda \in (0, 1)$. A differentiable function f is said to be c_0-*strongly convex* for a constant $c_0 > 0$, if

$$f(y) \ge f(x) + \nabla f(x)^T(y-x) + \frac{c_0}{2}\|y-x\|^2,\ \forall x, y \in \mathbb{R}^n. \tag{1}$$

For a twice continuously differentiable function f, it is c_0-strongly convex if and only if $\nabla^2 f(x) \ge c_0 I_n$.

Consider a multi-agent network described by a weighted graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the node set and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is the edge set. $\mathcal{A} = [a_{ij}]_{N\times N}$ is its adjacency matrix with $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ (meaning that agent j can send its information to agent i), and $a_{ij} = 0$, otherwise. A graph is said to be strongly connected if, for any pair of vertices, there exists a sequence of intermediate vertices connected by edges. For $i \in \mathcal{V}$, the weighted in-degree and out-degree are $d_{in}^i = \sum_{j=1}^N a_{ij}$ and $d_{out}^i = \sum_{j=1}^N a_{ji}$, respectively. A graph is weight-balanced if $\forall\, i \in \mathcal{V}$, $d_{in}^i = d_{out}^i$. The Laplacian matrix is $L = \mathcal{D}_{in} - \mathcal{A}$, where $\mathcal{D}_{in} = \text{diag}\{d_{in}^1, \dots, d_{in}^N\} \in \mathbb{R}^{N\times N}$. The following result is well known.

Lemma 1. *Graph $\mathcal{G}$ is weight-balanced if and only if $L + L^T$ is positive semidefinite; it is strongly connected only if zero is a simple eigenvalue of L.*

Distributed resource allocation optimization is usually formulated as follows. For each agent $i \in \mathcal{V}$, there are a local decision variable $x_i \in \mathbb{R}^n$ and a local cost function $f_i(x_i) : \mathbb{R}^n \to \mathbb{R}$. The agents cooperatively minimize the total cost function of the network, defined as $f(\boldsymbol{x}) \triangleq \sum_{i=1}^N f_i(x_i)$, subject to the resource allocation constraint $\sum_{i=1}^N x_i = \sum_{i=1}^N b_i = d$, that is,

$$\min_{\boldsymbol{x}\in\mathbb{R}^{nN}} f(\boldsymbol{x}),\ \text{s.t.}\ (1_N^T \otimes I_n)\boldsymbol{x} = d, \tag{2}$$

where $\boldsymbol{x} \triangleq col\{x_1, \dots, x_N\}$ and $d \in \mathbb{R}^n$.

The following assumption is adopted to ensure the well-posedness of (2), which is widely used.

Assumption 1.

(1) $f(\boldsymbol{x})$ is c_0-strongly convex for some constant $c_0 > 0$ and is twice continuously differentiable.
(2) The interaction graph $\mathcal{G}$ is strongly connected and weight-balanced.

The following lemma is quite fundamental for problem (2), whose proof is simple and omitted here.

Lemma 2. *Under Assumption 1, there exists a unique optimal solution $\boldsymbol{x}^* = col\{x_1^*, \dots, x_N^*\}$ of problem (2). In addition, there exists a unique $\boldsymbol{\lambda}^* = col\{\mu^*, \dots, \mu^*\}$ such that the following condition holds.*

$$\begin{cases} 0 = \nabla f(\boldsymbol{x}^*) + \boldsymbol{\lambda}^* \\ 0 = (1_N^T \otimes I_n)\boldsymbol{x}^* - d. \end{cases} \tag{3}$$

The goal of this paper is to design a distributed sub-optimal algorithm with a positive adjustable parameter ε for problem (2), such that

(1) the equilibrium point of the proposed algorithm is exponentially stable with the resource allocation constraint held;
(2) it approaches the optimal solution of problem (2) as $\varepsilon \to 0$, and the distance is bounded linearly by ε.

Of course, the design of sub-optimal algorithm should be simpler than those for optimal solutions.

3. Distributed algorithm design

In this section, we propose a distributed sub-optimal algorithm, and also show the relationship between its design and singular perturbation analysis.

To make a comparison, we first introduce a distributed algorithm over undirected graphs for problem (2), obtained in the literature, such as (Yi et al., 2016):

$$\forall\, i \in \mathcal{V},\ \begin{cases} \dot{x}_i = -\nabla f_i(x_i) - \lambda_i \\ \dot{\lambda}_i = -k_P \sum_{j=1}^N a_{ij}(\lambda_i - \lambda_j) \\ \qquad - k_I \sum_{j=1}^N a_{ij}(z_i - z_j) + x_i - b_i \\ \dot{z}_i = \sum_{j=1}^N a_{ij}(\lambda_i - \lambda_j) \end{cases} \tag{4}$$