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Technical communique

A fixed-time convergent algorithm for distributed convex optimization in multi-agent systems☆

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ABSTRACT

This technical paper presents a distributed continuous-time algorithm to solve multi-agent optimization problem with the team objective being the sum of all local convex objective functions while subject to an equality constraint. The optimal solutions are achieved within fixed time which is independent of the initial conditions of agents. This advantage makes it possible to off-line preassign the settling time according to task requirements. The fixed-time convergence for the proposed algorithm is rigorously proved with the aid of convex optimization and fixed-time Lyapunov theory. Finally, the algorithm is valuated via an example.

1. Introduction

Distributed optimization has received significant attention in recent years, due to its potential applications in sensor networks (Li & Elia, 2015; Yang, Chen, Wang, & Shi, 2014), power systems (Chen & Zhao, 2018; Cherukuri & Cortés, 2016; Yang, Tan, & Xu, 2013; Yi, Hong, & Liu, 2016), machine learning (Boyd, Parikh, Chu, Peleato, & Eckstein, 2011), and so on. Distributed optimization means that local agents cooperatively minimize a team objective function while subject to some constraints. Up to now, various distributed algorithms have been proposed to solve the optimization problems in multi-agent systems. Some algorithms employ the iterative calculation scheme (see, for example, distributed subgradient algorithms (Nedić & Ozdaglar, 2009; Nedić, Ozdaglar, & Parrilo, 2010; Zhu & Martínez, 2012), alternating direction method of multipliers (Boyd et al., 2011), Newton method (Wei, Ozdaglar, & Jadbabaie, 2013), gossip algorithm (Lu, Tang, Regier, & Bow, 2011), and gradient method (Guo, Wen, Mao, Li, & Song, 2017)). The other class of algorithms use the continuous time coordination scheme (see, for example, the continuous-time consensus-based optimization (Chen & Zhao, 2018; Gharesifard & Cortés, 2014; Kia, Cortés, & Martínez, 2015; Qiu, Liu, & Xie, 2016; Wang & Elia, 2011), continuous time zero-gradient-sum algorithms (Chen & Ren, 2016; Lu & Tang, 2012), subgradient projection algorithm (Lin, Ren, & Song, 2016), continuous time gradient algorithm (Yi, Hong, & Liu, 2015), initialization-free distributed optimization algorithms (Cherukuri & Cortés, 2016; Yi et al., 2016)).

All the aforementioned discrete time and continuous time algorithms are developed based on linear protocols, which reach the optimal solutions asymptotically or exponentially, that is, optimal solutions are achieved over an infinite-time horizon. This means that only suboptimal solutions are obtained in practical application. Therefore, it is highly desirable that the optimal solutions can be achieved in finite time (Bhat & Bernstein, 2000). As far as we know, only few works discussed the finite-time distributed optimization problem. Based on the discontinuous algorithms (Chen, Ren, & Feng, 2017; Lin, Ren, & Farrell, 2017; Pilloni, Pisano, Franceschelli, & Usai, 2016), the convex optimization problems can be solved in finite time. However, the discontinuous algorithms may be subject to chattering effect. Song and Chen (2016) present a continuous finite-time algorithm, where only the unconstrained convex optimization problem is discussed. The settling time in these finite-time optimization algorithms (Chen et al., 2017; Lin et al., 2017; Pilloni et al., 2016; Song & Chen, 2016) depends on the initial conditions of agents. It is hard to off-line preassign the settling time as the initial conditions of agents may be unavailable in advance. To solve this shortcoming, the fixed-time stability has been recently addressed (see Fu & Wang, 2017;

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Polyakov, 2012; Polyakov, Efimov, & Perruquetti, 2015; Zuo, 2015). However, no results on distributed fixed-time optimization have been proposed up to now.

In this technical paper, we solve a distributed convex optimization problem with equality constraint using continuous time nonlinear protocol. As pointed out in Pilloni et al. (2016), advances in speed and computing power of modern digital signal processors now permit the accurate implementation of algorithms designed in the continuous-time domain. Moreover, the continuous protocols can be implemented through physical propagation. The main contributions of this technical paper are summarized in the following two aspects. First, we present a distributed fixed-time optimization algorithm, which is able to off-line preassign the settling time according to task requirements. Second, the proposed algorithm is designed based on continuous function other than the discontinuous function (Chen et al., 2017; Lin et al., 2017; Pilloni et al., 2016), and thus the control input chattering phenomenon can be avoided.

Notations: Let $s = [s_1, s_2, \ldots, s_n]^T$ be an n dimensional column vector; s^T denotes the transpose of the vector s; $\|s\|$ is the 2-norm of the vector s. $\mathbf{1}$ denotes an n dimensional column vector of all ones. Let $sig(s)^\rho = [sig(s_1)^\rho, sig(s_2)^\rho, \ldots, sig(s_n)^\rho]^T$ with $sig(s_i)^\rho = sign(s_i)|s_i|^\rho\,(i = 1, 2, \ldots, n)$, where $sign(\cdot)$ represents the sign function and $|s_i|$ denotes the absolute value of s_i. $\nabla f(s)$ and $\nabla^2 f(s)$ respectively denote the gradient and the Hessian matrix of the function $f(s)$.

2. Preliminaries and problem formulation

2.1. Graph theory

The communication topology among agents is denoted by a weighted undirected graph $G = (V, E, A)$ with the node set $V = \{1, 2, \ldots, n\}$ denoting the agents and the edge set $E \subseteq V \times V$ representing the communication links. An undirected edge $(i, j) \in E$ indicates that agent i and agent j can exchange information with each other. The set of communication neighbors of agent i is denoted by $N_i = \{j|(j, i) \in E\}$. A path between agent i and agent j is a sequence of distinct edges of the form $(i, i_1), (i_1, i_2), \ldots, (i_s, j)$, where $i, i_1, \ldots, i_s, j$ are distinct agents. Graph G is said to be connected if there exists a path between any two agents. Define the weighted adjacency matrix $A = [a_{ij}] \in R^{n\times n}$ with $a_{ij} = a_{ij} > 0$ if $(i, j) \in E$ and $a_{ij} = 0$ otherwise. The Laplacian matrix of graph G is defined as $L = [l_{ij}] \in R^{n\times n}$ with $l_{ii} = \sum_{j\in N_i} a_{ij}$ and $l_{ij} = -a_{ij}, \forall i \neq j$. For an undirected and connected graph, 0 is a simple eigenvalue of the Laplacian matrix L with the eigenvector $\mathbf{1}$, and all the other eigenvalues are positive.

2.2. Several useful lemmas

Lemma 1 (*Zuo, 2015*). *Let $\zeta_1, \zeta_2, \ldots, \zeta_n \geq 0$. Then*

$$\sum_{i=1}^{n} \zeta_i^b \geq \left(\sum_{i=1}^{n} \zeta_i\right)^b, \text{ if } 0 < b \leq 1 \tag{1}$$

$$\sum_{i=1}^{n} \zeta_i^b \geq n^{1-b}\left(\sum_{i=1}^{n} \zeta_i\right)^b, \text{ if } 1 < b < \infty \tag{2}$$

Lemma 2 (*Polyakov, 2012*). *Consider the following system*

$$\dot{x} = f(t, x), x(0) = x_0 \tag{3}$$

where $x \in R^n$ and $f : R \times R^n \to R^n$ is a nonlinear function. Assume that the origin is the equilibrium point of the system (3). If there exists a continuous radially unbounded function $V : R^n \to R_+ \cup \{0\}$ such that (1) $V(x) = 0 \Leftrightarrow x = 0$; (2) any solution $x(t)$ of (3) satisfies the inequality $\dot{V}(x(t)) \leq -(\alpha V^p(x(t)) + \beta V^q(x(t)))^k$ for some $\alpha, \beta, p, q, k > 0 : pk < 1, qk > 1$. Then the origin of the system (3) is globally fixed-time stable, and the following estimate of the settling time holds

$$T(x_0) \leq \frac{1}{\alpha^k(1 - pk)} + \frac{1}{\beta^k(qk - 1)}, \forall x_0 \in R^n \tag{4}$$

2.3. Problem formulation

Consider a multi-agent system consisting of n agents, where each agent has a local objective function $C_i(x_i)$, which is only known by agent i. The objective of multi-agent optimization is to cooperatively minimize the sum of all local objective functions while maintaining a global equality constraint, i.e.,

$$\text{Min } C(x) = \sum_{i=1}^{n} C_i(x_i) \tag{5}$$

$$\text{s. t. } \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} X_{i0} = X \tag{6}$$

where $x = [x_1, x_2, \ldots, x_n]^T$ is a decision vector; $C(x)$ is a team objective function; X_{i0} denotes the initial value of the variable x_i; X is a global constraint variable.

Remark 1. The equality (6) comes from some physical constraints such as resource allocation, supply–demand balance, and lending-and-borrowing balance.

The following assumptions are required in our following analysis.

Assumption 1. The fixed communication topology among agents is undirected and connected.

Assumption 2. The objective functions $C_i(x_i), (i = 1, 2, \ldots, n)$ are strongly convex such that $\nabla^2 C_i(x_i) \geq \underline{\sigma} > 0$ for some positive constant $\underline{\sigma}$.

Assumption 2 implies that the optimization problem of (5) and (6) has a unique optimal solution.

3. Main results

In this section, the following distributed fixed-time algorithm is proposed to solve the optimization problem (5) under equality constraint (6).

$$\begin{cases} \dot{y}_i = -k_1 sig\left(\sum_{j\in N_i} a_{ij}(\psi_j - \psi_i)\right)^\mu - k_2 sig\left(\sum_{j\in N_i} a_{ij}(\psi_j - \psi_i)\right)^\nu \\ x_i = \sum_{j\in N_i} a_{ij}(y_j - y_i) + X_{i0} \\ \psi_i = \dfrac{\partial C_i(x_i)}{\partial x_i} \end{cases} \tag{7}$$

where k_1 and k_2 are positive constants; the constants μ and ν satisfy $0 < \mu < 1$ and $\nu > 1$, respectively; y_i and ψ_i are auxiliary variables. From (7), we see that the neighboring agents need to exchange the information of the auxiliary variables y_j and $\psi_j, j \in N_i$.

Theorem 1. *Under Assumptions 1 and 2, the distributed algorithm (7) solves the optimization problem of (5) and (6) in fixed time.*