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Mild solutions to the dynamic programming equation for stochastic optimal control problems☆

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ABSTRACT

We show via the nonlinear semigroup theory in $L^1(\mathbb{R})$ that the 1-D dynamic programming equation associated with a stochastic optimal control problem with multiplicative noise has a unique mild solution in a sense to be made precise.

1. Introduction

Consider the following stochastic optimal control problem

$$\underline{\text{Minimize}} \quad \mathbb{E}\left\{\int_0^T \Big(g\big(X(t)\big)+h\big(u(t)\big)\Big)\,dt+g_0\big(X(T)\big)\right\}, \tag{1}$$

subject to $u \in \mathcal{U}$ and to state equation

$$\begin{cases} dX = f(X)\,dt+\sqrt{u}\,\sigma(X)\,dW, & \text{for } t\in(0,T) \\ X(0)=X_0 \end{cases} \tag{2}$$

where $\mathcal{U}$ is the set of all $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $u : (0,T) \to \mathbb{R}^+ = [0,+\infty]$ and $W : \mathbb{R} \to \mathbb{R}$ is an 1-D Wiener process in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, provided the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Here $X_0 \in \mathbb{R}$, while $X : [0,T] \to \mathbb{R}$ is the strong solution to (2). We would like to underline that the studied optimization problem is related to the so called stochastic volatility models, used in the financial framework, whose relevance has raised exponentially during last years. In fact such models, contrarily to the constant volatility ones as, e.g., the standard Black and Scholes approach, the Vasicek interest rate model, or the Cox–Ross–Rubinstein model, allow to consider the more realistic situation of volatility levels changing in time. As an example, the latter is the case of the Heston model, see Heston (1993), where the variance is assumed to be a stochastic process following a Cox–Ingersoll–Ross (CIR) dynamic, see Cordoni and Di Persio (2013) or Cox, Ingersoll and Ross (1985) and references therein for more recent related techniques, as well as the case of the Constant Elasticity of Variance (CEV) model, see Cox (1975), where the volatility is expressed by a power of the underlying level, which is often referred as a local stochastic volatility model. Other interesting examples, which is the object of our ongoing research particularly from the numerical point of view, include the Stochastic Alpha, Beta, Rho (SABR) model, see, e.g., Hagan, Lesniewski, and Woodward (2015), and models which are used to estimate the stochastic volatility by exploiting directly markets data, as happens using the GARCH approach and its variants. Within latter frameworks and due to several macroeconomic crises that have affected different (type of) financial markets worldwide, governments decided to become *active players of the game*, as, e.g., in the recent case of the *Volatility Control Mechanism* (VCM) established for the securities, resp. for the derivatives, market established in August 2016, resp. in January 2017, within the Hong Kong Stock Exchange (HKEX) framework, see, e.g., Stein (2006) and Stein (2012) and references therein for other applications and examples. It should be said however that problems of the form (1)–(2) are relevant in other applications as well.

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Hypotheses:

(i) $h : \mathbb{R} \to \mathbb{R}$ is convex, continuous and $h(u) \ge \alpha_1 |u|^2, \forall u \in \mathbb{R}$, for some $\alpha_1 > 0$.
(ii) $f \in C_b^2(\mathbb{R}), f'' \in L^1(\mathbb{R}), g, g_0 \in W^{2,\infty}(\mathbb{R})$.
(iii) $\sigma \in C_b^1(\mathbb{R})$, and

$$|\sigma(x)| \ge \rho > 0, \quad \forall x \in \mathbb{R}. \tag{3}$$

In the following H^* is the Legendre conjugate of function

$$H(u) = h(u) + I_{[0,\infty)}(u) = \begin{cases} h(u) & \text{if } u \ge 0\,, \\ +\infty & \text{otherwise.} \end{cases}$$

Namely,

$$H^*(p) = \sup\{p\,u - H(u) \,:\, u \in \mathbb{R}\}, \quad \forall p \in \mathbb{R}. \tag{4}$$

We have $\partial H^*(p) = (\partial h + N_{[0,\infty)})^{-1} p$, where ∂ is the subdifferential symbol, and $N_{[0,\infty)}$ is the normal cone to $[0,\infty)$. This yields

$$0 \le \partial H^*(p) \le C(|p| + 1)\,, \ \forall p \in \mathbb{R}\,. \tag{5}$$

We denote also by j the potential of H^*, that is

$$j(r) = \int_0^r H^*(p)\, dp, \quad \forall r \in \mathbb{R}.$$

The dynamic programming equation corresponding to the stochastic optimal control problem (1) is given by (see, e.g., Fleming & Rishel, 2012; Øksendal, 2003),

$$\begin{cases} \varphi_t(t,x) + \min\limits_u \left\{\frac{1}{2}\sigma^2\, \varphi_{xx}(t,x)\, u + H(u)\right\} \\ \qquad + f(x)\, \varphi_x(t,x) + g(x) = 0, \quad \forall t \in [0,T], x \in \mathbb{R} \\ \varphi(T,x) = g_0(x), \quad x \in \mathbb{R}, \end{cases}$$

or equivalently

$$\begin{cases} \varphi_t(t,x) - H^*\big(-\frac{1}{2}\sigma^2\, \varphi_{xx}(t,x)\big) + f(x)\, \varphi_x(t,x) \\ \qquad\qquad + g(x) = 0, \quad \forall (t,x) \in [0,T] \times \mathbb{R} \\ \varphi(T,x) = g_0(x), \quad x \in \mathbb{R}\,. \end{cases} \tag{6}$$

Moreover, if φ is a smooth solution to (1) the associated feedback controller

$$u(t) = \arg\min_u \left\{ \frac{1}{2}\, \sigma^2\, \varphi_{xx}\big(t, X(t)\big)\, u + H(u) \right\}, \tag{7}$$

is optimal for problem (1).

We set $\psi = \varphi_x$ and replace Eq. (6) by

$$\begin{cases} \psi_t(t,x) - \big(H^*\big(-\frac{1}{2}\sigma^2\, \psi_x(t,x)\big)\big)_x + (f(x)\, \psi(t,x))_x \\ \qquad\qquad + g'(x) = 0, \quad \forall (t,x) \in [0,T] \times \mathbb{R} \\ \psi(T,x) = g_0'(x), \quad x \in \mathbb{R}\,. \end{cases} \tag{8}$$

Up to our knowledge, in literature the rigorous treatment of existence theory for equations of this form has been shown so far within the theory of viscosity solutions only. (See, e.g., Crandall, Ishii, & Lions, 1992.) However, the known existence results for viscosity solutions are not directly applicable in the present case. Here we shall exploit a different approach, namely we use a suitable transformation aiming at reducing (8) to a one dimensional Fokker–Planck equation which is then treated as a nonlinear Cauchy problem in $L^1(\mathbb{R})$. As regards the non-degenerate hypothesis (3) it will be later on dispensed by assuming more regularity on function σ. (See Section 4.)

1.1. Notation and basic results

We shall use the standard notation for functional spaces on $\mathbb{R}$. In particular $C_b^k(\mathbb{R})$ is the space of functions $y : \mathbb{R} \to \mathbb{R}$, differentiable of order k and with bounded derivatives until order k. By $L^p(\mathbb{R})$, $1 \le p \le \infty$, we denote the classical space of Lebesgue-measurable p-integrable functions on $\mathbb{R}$ with the norm $\|\cdot\|_p$ and by $H^k(\mathbb{R}^n)$, $W^{k,p}(\mathbb{R}^n)$, $k = 1,2$, the standard Sobolev spaces on $\mathbb{R}^n$, $n = 1,2$. Denote by $\langle \cdot,\cdot \rangle_2$ the scalar product of $L^2(\mathbb{R})$. We set also $y_x = y' = \partial y/\partial x$, $y_t = \partial y/\partial t$, $y_{xx} = \partial^2 y/\partial x^2$, for $x \in \mathbb{R}$. By $\mathcal{D}'(\mathbb{R}^n)$ we denote the space of Schwartz distributions on $\mathbb{R}^n$.

Definition 1.1 (*Accretive Operator*). Given a Banach space X, a nonlinear operator A from X to itself, with domain $D(A)$, is said to be *accretive* if $\forall u_i \in D(A), \forall v_i \in A\,u_i$, $i = 1,2$, there exists $\eta \in J(u_1 - u_2)$ such that

$${}_X\langle v_1 - v_2, \eta \rangle_{X'} \ge 0\,, \tag{9}$$

where X' is the dual space of X, ${}_X\langle\cdot,\cdot\rangle_{X'}$ is the duality pairing and $J : X \to X'$ is the *duality mapping* of X. (See, e.g., Barbu, 2010.)

An accretive operator A is said to be *m-accretive* if $R(\lambda I + A) = X$ for all (equivalently some) $\lambda > 0$, while it is said to be *quasi-m-accretive* if there is $\lambda_0 \in \mathbb{R}$ such that $\lambda_0 I + A$ is m-accretive.

We refer to Barbu (2010) for basic results on m-accretive operators in Banach spaces and the corresponding associated Cauchy problem.

2. Existence results

We set

$$y(t,x) = -\psi_x(T-t,x), \quad \forall t \in [0,T], x \in \mathbb{R}, \tag{10}$$

and we rewrite Eq. (8) as

$$\begin{cases} y_t(t,x) - \Big(H^*\big(\frac{\sigma^2}{2}\, y(t,x)\big)\Big)_{xx} - f''(x)\psi(T-t,x) \\ \qquad - 2f'(x)\psi_x(T-t,x) - f(x)\psi_{xx}(T-t,x) = -g''(x), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,T) \times \mathbb{R} \\ y(0,x) = -g_0''(x), \quad x \in \mathbb{R}. \end{cases} \tag{11}$$

We set

$$\Phi(z)(x) = \int_{-\infty}^x z(\xi)\, d\xi\,, \quad z \in L^1(\mathbb{R}). \tag{12}$$

Then by (10) we have

$$\psi(t,x) = -\Phi\big(y(T-t,x)\big)\,, \quad \forall t \in [0,T]. \tag{13}$$

Setting

$$B\,y = -f''\Phi(y) - 2f'y, \quad \forall y \in L^1(\mathbb{R})\,, \tag{14}$$

and taking into account that $f' \in L^\infty(\mathbb{R}), f'' \in L^1(\mathbb{R})$, we obtain for the operator B the estimate

$$\|B\,y\|_1 \le C\, \|y\|_1, \quad \forall y \in L^1(\mathbb{R})\,. \tag{15}$$

Therefore Eq. (11) can be rewritten as follows

$$\begin{cases} y_t - \Big(H^*\big(\frac{\sigma^2}{2}\, y\big)\Big)_{xx} + f\, y_x + B\,y = g_1, \quad \text{in } [0,T] \times \mathbb{R} \\ y(0) = y_0 \in L^1(\mathbb{R}) \end{cases}, \tag{16}$$

where $y_0 = -g_0''$ and $g_1 = -g''$ in $\mathcal{D}'(\mathbb{R})$.

Definition 2.1. The function $y : [0,T] \times \mathbb{R} \to \mathbb{R}$ is said to be a *mild* solution to Eq. (16) if $y \in C([0,T]; L^1(\mathbb{R}))$ and

$$y(t) = \lim_{\epsilon \to 0} y_\epsilon(t) \text{ in } L^1(\mathbb{R}), \quad \forall t \in [0,T]\,, \tag{17}$$

$$y_\epsilon(t) = y_\epsilon^i, \ \text{ for } t \in [i\epsilon, (i+1)\epsilon], \ i = 0,1,\ldots, \left[\frac{T}{\epsilon}\right] - 1 = \left[\frac{T}{\epsilon}\right], \tag{18}$$