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Stability of positive coupled differential-difference equations with unbounded time-varying delays☆

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ABSTRACT

This paper considers the stability problem of a class of positive coupled differential-difference equations with unbounded time-varying delays. A new method, which is based on upper bounding of the state vector by a decreasing function, is presented to analyze the stability of the system. Different from the existing methods, our method does not use the usual Lyapunov–Krasovskii functional method or the comparison method based on positive systems with constant delays. A new criterion is derived which ensures asymptotic stability of the system with unbounded time-varying delays. A numerical example with simulation results is given to illustrate the stability criterion.

1. Introduction

Many systems in engineering are described by a differential equation coupled with a difference equation (see Gu, 2010; Mazenc, Ito, & Pepe, 2013; Pepe, Jiang, & Fridman, 2008 and the references therein). Such equations are called coupled differential-difference equations (CDDEs). They cover many important classes of dynamical systems, such as neutral systems, systems with multiple commensurate delays, some singular systems as special cases (Gu & Liu, 2009; Gu & Niculescu, 2006). Many partial differential equations with nonstandard derivative boundary conditions, for example, lossless propagation systems described by a partial differential equation of hyperbolic type, can also be reformulated into CDDEs with finite delay (Niculescu, 2001; Răsvan, 2006). The most widely used approach to analyze stability of CDDEs is based on the discretized Lyapunov–Krasovskii functional method combined with linear matrix inequalities (LMIs) (see Gu, 2010; Gu & Liu, 2009; Gu, Zhang, & Xu, 2011; Karafyllis, Pepe, & Jiang, 2009; Li & Gu, 2010; Mazenc et al., 2013; Pepe et al., 2008; Zhang, Peet, & Gu, 2011 and the references therein).

Positive systems, whose states are never negative whenever the initial conditions are non-negative, appear naturally in many fields such as biology, industrial engineering and economics (see Kaczorek, 2002). The stability problem of positive systems with time delays has also attracted significant research attention in recent years (see Ait Rami, 2009; Liu & Dang, 2011; Liu, Yu, & Wang, 2010; Nam, Phat, Pathirana, & Trinh, 2016; Nam, Trinh, & Pathirana, 2016; Ngoc, 2013; Ngoc & Trinh, 2016; Phat & Sau, 2014; Shen & Lam, 2014; Zhu, Li, & Zhang, 2012). Recently, an overview on the recent developments of stability of linear positive time-delay systems has been given in Briat (2017). Shen and Zheng (2015) considered the stability problem of a class of positive CDDEs with bounded time-varying delay. Their method is based on a comparison between the solution of CDDEs with a time-varying delay and the solution of CDDEs with a constant delay, which is an upper bound of the time-varying delay. Hence, it is not possible to extend this method to CDDEs with infinity time-varying delays.

Normally, time delays which appear in engineering systems, are bounded (see Fridman, 2014; Gu, Chen, & Kharitonov, 2003; Shafai & Sadaka, 2012; Shafai, Sadaka, & Ghadami, 2012; Sipahi, Vyhlidal, & Niculescu, 2012). However, it also appears that there are many dynamical systems whose time delays are unbounded. In dynamical systems that have a spacial nature, such as neural networks, time delay is often unbounded. Therefore, in recent years there has been a growing interest in the stability problem of dynamical systems with unbounded time-delay (see Chen & Liu, 2017; Feyzmahdavian, Charalambous, & Johansson, 2014; Li & Cao, 2017; Liu & Dang, 2011; Liu, Lu, & Chen, 2010; Shen & Lam, 2015; Zhou, 2013, 2014).

So far, most of the existing results reported only on the stability of CDDEs with constant time delays or bounded time-varying delays. No papers have reported on the stability of CDDEs with unbounded time-varying delays. In this paper, inspired by the work of Liu and Dang (2011), we study the stability problem of

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a class of positive CDDEs with unbounded time-varying delays. Instead of using the comparison method (Shen & Zheng, 2015), we construct estimates of the solution of the considered CDDEs on non-equal time sub-intervals. As a result, we can analyze the stability of CDDEs with unbounded time-varying delays. A new stability condition for the system is obtained. The obtained result is illustrated by a numerical example.

2. Notations and a problem statement

Notations. $\mathbb{R}^n(\mathbb{R}^n_{0,+}, \mathbb{R}^n_+)$ is the n-dimensional (nonnegative, positive) vector space; $\overline{1,n} = \{1, 2, \ldots, n\}$; given two vectors $x = [x_1\, x_2\, \cdots\, x_n]^T \in \mathbb{R}^n$, $y = [y_1\, y_2\, \cdots\, y_n]^T \in \mathbb{R}^n$, two $n \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, the following notations will be used in our development: $x \prec y(\preceq y)$ means that $x_i < y_i(\leq y_i)$, $\forall i \in \overline{1,n}$; $A \prec B(\preceq B)$ means that $a_{ij} < b_{ij}(\leq b_{ij})$, $\forall i, j \in \overline{1,n}$; A is nonnegative if $0 \preceq A$; A is a Metzler matrix if $a_{ij} \geq 0$, $\forall i, j \in \overline{1,n}, i \neq j$; $\|x\|_\infty = \max_{i=1}^n |x_i|$; $s(A) = \max\{Re(\lambda) : \lambda \in \sigma(A)\}$ stands for the spectral abscissa of a matrix A; $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ stands for the spectral radius of a matrix A. A is a Schur matrix if $\rho(A) < 1$; The limitation of a vector-valued function is understood in the component-wise sense.

Consider the following linear CDDEs with time-varying delays

$$\dot{x}(t) = Ax(t) + By(t - \tau(t)),\ t \geq 0, \tag{1}$$

$$y(t) = Cx(t) + Dy(t - h(t)), \tag{2}$$

where $x(.) \in \mathbb{R}^n$, $y(.) \in \mathbb{R}^m$ are the state vectors. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$ are known constant matrices. D is assumed to be a Schur matrix. Here, the delays $\tau(.) \in \mathbb{R}_{0,+}$ and $h(.) \in \mathbb{R}_{0,+}$ are unknown time-varying delays. Similar to Liu and Dang (2011), the delays can be unbounded and assumed to satisfy the following growth condition:

Assumption 1. There exist a positive scalar $T > 0$ and a scalar $\theta \in (0, 1)$ such that

$$\max\left\{\sup_{t\geq T} \frac{\tau(t)}{t}, \sup_{t\geq T} \frac{h(t)}{t}\right\} \leq \theta. \tag{3}$$

It is easy to see that all the bounded delays satisfy condition (3). Furthermore, condition (3) implies that $t - \tau(t) \geq (1 - \theta)t > 0$ and $t - h(t) \geq (1 - \theta)t > 0$ for all $t \geq T$. Hence, the initial condition of system (1)–(2) is given by $x(0) = \psi(0)$, $y(s) = \phi(s)$, $s \in [-\max_{t\in[0,T]} \max h(t), \tau(t), 0)$. Let us denote by $x(t, \psi, \phi)$ and $y(t, \psi, \phi)$ the state trajectories with the initial condition (ψ, ϕ) of system (1)–(2).

The main objective of this paper is to derive a sufficient condition for the asymptotic stability of system (1)–(2).

3. Main result

The following lemmas are needed for our development.

Lemma 1 (Berman & Plemmons, 1994). *(i) Let $M \in \mathbb{R}^{n\times n}_+$ be a nonnegative matrix. Then, the following statements are equivalent: (i_1) M is Schur stable; (i_2) $(M - I)q \prec 0$ for some $q \in \mathbb{R}^n_+$; (i_3) $(I - M)^{-1} \succeq 0$.*

(ii) Let $M \in \mathbb{R}^{n\times n}$ be a Metzler matrix. Then, the following statements are equivalent: (ii_1) M is Hurwitz stable; (ii_2) $Mq \prec 0$ for some $q \in \mathbb{R}^n_+$; (ii_3) $M^{-1} \preceq 0$.

Definition 2 (Kaczorek, 2002). System (1)–(2) is said to be positive if for any non-negative initial values, $\psi(0) \succeq 0$, $\phi(s) \succeq 0$, $s \in [-T, 0)$, the state trajectories of system (1)–(2) satisfy that $x(t, \psi, \phi) \succeq 0$, $\forall t \geq 0$ and $y(t, \psi, \phi) \succeq 0$, $\forall t \geq 0$.

Lemma 3 (Ngoc & Trinh, 2016; Shen & Zheng, 2015). *Assume that A is a Metzler matrix, B, C, D are nonnegative, D is a Schur matrix. Then,*

(i) For all piecewise continuous functions $\omega(t) \succeq 0$, $u(t) \succeq 0$, the following system is positive

$$\dot{x}(t) = Ax(t) + By(t - \tau(t)) + \omega(t), \tag{4}$$

$$y(t) = Cx(t) + Dy(t - h(t)) + u(t). \tag{5}$$

(ii) For $\psi_1(0) \preceq \psi_2(0)$ and $\phi_1(s) \preceq \phi_2(s)$, $s \in [-T, 0)$, we have

$$x(t, \psi_1, \phi_1) \preceq x(t, \psi_2, \phi_2),\ \forall t \geq 0, \tag{6}$$

$$y(t, \psi_1, \phi_1) \preceq y(t, \psi_2, \phi_2),\ \forall t \geq 0. \tag{7}$$

(iii) Assume that $s(A + B(I - D)^{-1}C) < 0$. Then, there exist $p \in \mathbb{R}^n_+$, $q \in \mathbb{R}^m_+$ and $\mu \in (0, 1)$ such that

$$Ap + Bq \prec 0, \tag{8}$$

$$Cp + Dq \prec (1 - \mu)q, \tag{9}$$

$$(I - D)^{-1}Cp \prec (1 - \mu)q. \tag{10}$$

Proof. The proof of (i) is similar to Shen and Zheng (2015). Therefore, we omit it here. We now use (i) to prove (ii). Since system (1)–(2) is linear, we have

$$x(t, \psi_2, \phi_2) - x(t, \psi_1, \phi_1) = x(t, \psi_2 - \psi_1, \phi_2 - \phi_1). \tag{11}$$

By the positivity of system (1)–(2), we have

$$x(t, \psi_2 - \psi_1, \phi_2 - \phi_1) \succeq 0,\ \forall t \geq 0. \tag{12}$$

From (11) and (12), we obtain (6). Similarly, we also obtain (7).

(iii) By Ngoc and Trinh (2016), there exist two vectors $p \in \mathbb{R}^n_+$ and $q \in \mathbb{R}^m_+$ such that

$$Ap + Bq \prec 0, \tag{13}$$

$$Cp + (D - I)q \prec 0. \tag{14}$$

Since D is Schur stable and nonnegative, $(I - D)^{-1}$ is nonnegative by Lemma 1. Left multiplying $(I - D)^{-1}$ on (14), we obtain

$$(I - D)^{-1}Cp - q \prec 0. \tag{15}$$

Since (14) and (15) are strict inequalities, there is a scalar $\mu \in (0, 1)$ such that inequalities (9) and (10) hold. The proof of Lemma 3 is completed. □

We are now in a position to introduce the main result in the form of the following theorem.

Theorem 4. *Assume that A is a Metzler matrix, B, C, D are nonnegative, D is a Schur matrix and $s(A + B(I - D)^{-1}C) < 0$. Then, for all time-varying delays satisfying condition (3), system (1)–(2) is asymptotically stable.*

Proof. *Step 1:* In this step, we prove that there exist a scalar $\mu^* \in (0, \mu)$ and a time $t_1 > 0$ such that

$$x(t, p, q) \preceq (1 - \mu^*)p,\ \forall t \geq t_1, \tag{16}$$

$$y(t, p, q) \preceq (1 - \mu^*)q,\ \forall t \geq t_1, \tag{17}$$

where $p, q\ \mu$ are defined in Lemma 3. Firstly, similar to Shen and Zheng (2015), we define $e_x(t) = p - x(t, p, q)$, $e_y(t) = q - y(t, p, q)$. Then, $e_x(t)$ and $e_y(t)$ satisfy the following system:

$$\dot{e}_x(t) = Ae_x(t) + Be_y(t - \tau(t)) - (Ap + Bq), \tag{18}$$

$$e_y(t) = Ce_x(t) + De_y(t - h(t)) - (Cp + (D - I)q). \tag{19}$$