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Brief paper

Cartesian product-based hierarchical scheme for multi-agent systems[☆]

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ABSTRACT

In this paper, we solve the average-consensus problem using a hierarchical scheme based on Cartesian product of strongly connected balanced graphs — an algebraic approach to design complex networks. We show that the Cartesian product based hierarchical scheme for multi-agent systems outperforms the single-layer control strategies for average-consensus problem in terms of convergence rate, and also the system matrix produced by Cartesian product-based hierarchical (CPH) scheme do not necessarily exhibit block circulant symmetry. We analyze that if the factors graphs in Cartesian product are cyclic pursuit graphs, then the CPH scheme provides the same convergence rate while requiring the same communication links as in the hierarchical cyclic pursuit (HCP). We provide simulation results to demonstrate the key theoretical results.

Distributed control of multi-agent systems has been an intensive research topic due to extensive range of potential applications such as distributed sensor networks (Kar & Moura, 2009), and cooperative control of unmanned air vehicles (UAV) (Olfati-Saber, Fax, & Murray, 2007). The single-layer distributed control (SLC) strategies to solve consensus problem (Galloway, Justh, & Krishnaprasad, 2013, 2016; Lin, Broucke, & Francis, 2004; Sinha & Ghose, 2006) are scalable. However, increasing the number of agents overwhelmingly decreases the convergence rate of multi-agent systems designed under SLC strategies (Smith, Broucke, & Francis, 2005). A hierarchical structure of multi-agent systems is an approach to improve the convergence rate (Mukherjee & Ghose, 2016; Shimizu & Hara, 2008; Smith et al., 2005; Tsubakino & Hara, 2012). However, these hierarchical schemes assume that the system matrix must be block circulant, which is a strong condition in many social networks, biological network, and engineered networks.

A Cartesian product of graphs could be a promising method to design a hierarchical system while requiring a control strategy at the first layer only. Cartesian product of graphs is a method to produce composite graphs or layered systems using prime graphs (Chapman, Nabi-Abdolyousefi, & Mesbahi, 2014; Christopher & Royle, 2001; Hammack, Imrich, & Klavžar, 2011). Cartesian product of graphs retains some important properties of the factor graphs to provide an algebraic method of designing a hierarchical system while reducing the analysis to individual graphs involved in the product. Recently, it has been proved that a consensus protocol is a low pass graph filter (Izumi, Azuma, & Sugie, 2016). Cartesian product of graphs, on the other hand, can be used to improve data storage, memory access, and also help improve algorithms by modularizing the filter computation and Fourier transform on graphs (Sandryhaila & Moura, 2014). Therefore, we say that the CPH scheme presented in this paper will find potential applications in signal processing on graphs.

In spite of the detailed studies on hierarchical structure for multi-agents systems (Iqbal, Leth, & Ngo, 2016; Shimizu & Hara, 2008; Smith et al., 2005; Tsubakino & Hara, 2012; Tsubakino, Yoshioka, & Hara, 2013), to the best of our knowledge, a hierarchical method for multi-agent systems under the Cartesian product of graphs has not been studied. The salient features of the Cartesian product based hierarchical (CPH) scheme proposed in this paper are the following:

- The CPH strategy solves the average consensus problem; that is, the convergence point in CPH scheme is the average of initial state values of all the agents.
- The CPH strategy provides an algebraic method to develop a hierarchical scheme. Cartesian product of factor graphs retains the properties of the factor graphs so that it significantly simplifies the system analysis.

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- The system matrix due to CPH scheme does not necessarily exhibit block circulant symmetry. However, one can always produces block circulant symmetry by reversing the order of factor digraphs involve in Cartesian product of digraphs. Thus, HCP becomes a special case of the CPH scheme.

The contributions of this paper are two-fold:

- We develop the CPH scheme to improve the convergence rate of the SLC method for multi-agent systems.
- We show that if the factor graphs in the Cartesian product are cyclic pursuit graphs, then the convergence rate of CPH strategy and the hierarchical cyclic pursuit (HCP) method are similar using the same number of communication links.

1. Preliminaries

1.1. Graph theory

Consider a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consisting of a non-empty node set $\mathcal{V} = \{1, 2, \ldots, n\}$, an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and an adjacency matrix $Q = [q_{ij}]$, where $q_{ij} = 0$ if there is no directed link from node i to node j, and $q_{ij} = 1$ if there is a directed link from node i to node j. The diagonal entries of the adjacency matrix are always zero for a loopless network. The Laplacian $\mathcal{L} = \mathcal{L}(\mathcal{G})$ of the graph is

$$\mathcal{L} = D - Q, \tag{1}$$

where D is a diagonal matrix with diagonal entries $d_i(\mathcal{G})$, denoting the out-degree of the ith node with $i = 1, 2, \cdots, n$. The eigenvalues of $\mathcal{L}$ are denoted by $\lambda_1 = 0, \lambda_2, \cdots, \lambda_n$, where $\Re\{\lambda_i\} \le \Re\{\lambda_{i+1}\}, \ \forall i \in \{1, 2, \ldots, n\}$.

A digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is strongly connected if there exist a directed path between every pair of nodes, and balanced if

$$\sum_j q_{ij} = \sum_j q_{ji}, \quad \forall\, i. \tag{2}$$

In the sequel, 1_n is used to denote the $n \times 1$ vector containing ones, and I_n is used to denote the $n \times n$ identity matrix.

1.2. Cartesian product of directed graphs

Let $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ be two directed graphs. The Cartesian product of the factor digraphs $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted by $\mathcal{G}_1 \Box \mathcal{G}_2$, is a graph $\mathcal{G}$ with vertex set $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$, and there is an edge from vertex (i, p) to vertex (j, q) in $\mathcal{V}$ if and only if either $i = j$ and $(p, q) \in \mathcal{E}_2$, or $p = q$ and $(i, j) \in \mathcal{E}_1$ (Chapman, Nabi-Abdolyousefi, & Mesbahi, 2012). An example is given in Fig. 1.

The Cartesian product of graphs is associative; that is, $(\mathcal{G}_1 \Box \mathcal{G}_2) \Box \mathcal{G}_3 = \mathcal{G}_1 \Box (\mathcal{G}_2 \Box \mathcal{G}_3)$.

2. Cartesian product-based hierarchical scheme for multi-agent systems

2.1. Two-layer hierarchical system based on Cartesian product

Consider the strongly connected balanced graphs $\mathcal{G}_{n_1}$ and $\mathcal{G}_{n_2}$, where n_1 is the number of agents in a group, and n_2 is the number of groups in the two-layer CPH strategy. The Laplacian matrix associated with the digraph $\mathcal{G} = \mathcal{G}_{n_1} \Box \mathcal{G}_{n_2}$ can be written as

$$\begin{aligned} \mathcal{L}(\mathcal{G}) &= \mathcal{L}_1 \otimes I_{n_2} + I_{n_1} \otimes \mathcal{L}_2, \\ &= \mathcal{L}_1 \oplus \mathcal{L}_2, \end{aligned} \tag{3}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are the Laplacian matrices associated with $\mathcal{G}_{n_1}$ and $\mathcal{G}_{n_2}$, respectively. In (3), $\otimes$ denote Kronecker product and $\oplus$ Kronecker sum, see e.g., Chapman et al. (2014) (for Cartesian product of strongly connected balanced graphs see Fig. 2).

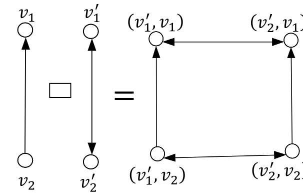
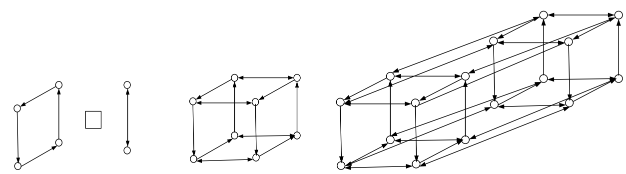


Fig. 1. Cartesian product of directed graphs.



(a) $\mathcal{G}_{n_1} \Box \mathcal{G}_{n_2} = \mathcal{G}$. (b) $\mathcal{G}_{n_1} \Box \mathcal{G}_{n_2} \Box \mathcal{G}_{n_3} = \mathcal{G}$.

Fig. 2. Cartesian product of cyclic pursuit graphs: (a) Cartesian product of $\mathcal{G}_{n_1}$ and $\mathcal{G}_{n_2}$ produces the third graph $\mathcal{G}$, (b) Cartesian product of $\mathcal{G}_{n_1}$, $\mathcal{G}_{n_2}$ and $\mathcal{G}_{n_3}$, where $\mathcal{G}_{n_3} = \mathcal{G}_{n_2}$ produces a three-layer hierarchy.

Remark 1. Matrix $\mathcal{L}_1$ represents how agents in groups interact with each others, thus by considering $\mathcal{L}_1$ as a block matrix in $\mathcal{L}$ given in (3) we see that $\mathcal{L}(\mathcal{G})$ does not necessarily exhibit block circulant symmetry. However, the hierarchical schemes presented in the literature are restricted to produce a system matrix with block circulant symmetry.

Remark 2. If one wishes to have a block circulant symmetry (with respect to $\mathcal{L}_1$ as block matrix) out of Cartesian products of digraph, then $\mathcal{G}_{n_2} \Box \mathcal{G}_{n_1}$ can be used, as $\mathcal{L}(\mathcal{G}_{n_2} \Box \mathcal{G}_{n_1}) = \mathcal{L}_2 \otimes I_{n_1} + I_{n_2} \otimes \mathcal{L}_1$.

The state equation of the multi-agent systems in the two-layer CPH scheme can be written as:

$$\dot{\tilde{x}} = M\tilde{x}, \tag{4}$$

where $M = -\mathcal{L}$, and
$\tilde{x} = (x_{1,1}, \ldots, x_{n_1,1}; x_{1,2}, \ldots, x_{n_1,2}; \cdots; x_{1,n_2}, \ldots, x_{n_1,n_2})$
is $N \times 1$ state vector, with $N = n_1 n_2$.

Convergence of solutions to (4) depends on the eigenvalues of M. According to Lemma 3.23 in Mesbahi and Egerstedt (2010), the eigenvalues of M are the elements in the set

$$S = \left\{ -\lambda_{i_1}^{\mathcal{L}_1} - \lambda_{i_2}^{\mathcal{L}_2} \,|\, 1 \le i_1 \le n_1, \ 1 \le i_2 \le n_2 \right\}, \tag{5}$$

where $\lambda_{i_1}^{\mathcal{L}_1}$ is the i_1th eigenvalue of $\mathcal{L}_1$, and $\lambda_{i_2}^{\mathcal{L}_2}$ is the i_2th eigenvalue of $\mathcal{L}_2$. Since $\mathcal{G}_{n_i}, i = 1, 2$ is a strongly connected balance graph, it follows from Theorem 4 in Olfati-Saber and Murray (2004) that the Laplacian matrix $\mathcal{L}_i$ associated with $\mathcal{G}_{n_i}$ has only positive eigenvalues and a simple zero eigenvalue. Thus, from (5) we conclude that all the eigenvalues of M in (4) lies in the open left-half plane, except for a simple zero eigenvalue, and that solutions to (4) convergence to a point for all initial conditions.

In the following, we prove that the two-layer CPH scheme solves the average consensus problem if each of $\mathcal{G}_{n_1}$ and $\mathcal{G}_{n_2}$ individually solves the average consensus problem. In the subsequent discussion, we use $\mathcal{N}(\mathcal{L}_1)$ to denote the *null space* of $\mathcal{L}_1$.

Since $\mathcal{G}_{n_1}$ is strongly connected balance graph, $\mathcal{L}_1 b$, where $b \in \mathbb{R}^{n_1}$, is always orthogonal to $1_{n_1}^T$, because $1_{n_1}^T \mathcal{L}_1 = 0$.

However, for an unbalanced graph $\mathcal{G}_1$ with n_1 nodes, we can choose $b \notin \mathcal{N}(\mathcal{L}_1)$, where $\mathcal{L}_1 = \mathcal{L}_1(\mathcal{G}_1)$, such that $\mathcal{L}_1 b$ is not orthogonal to $1_{n_1}^T$. For example, suppose that the kth entry of $1_{n_1}^T \mathcal{L}_1$ is *nonzero*. Choosing b such that the entries $b_i = 0$ for all $i \neq k$ and $b_k \neq 0$, then we have $1_{n_1}^T \mathcal{L}_1 b \neq 0$.