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Brief paper

On robustness analysis of linear vibrational control systems☆

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ABSTRACT

By injecting high frequency dither signals, it is possible to stabilize an inverted pendulum without any feedback. The concept of the vibrational control system is thus proposed to provide extra design freedom in stabilization or other performance indexes. Although various vibrational control algorithms have been proposed and implemented in literature, little work has been done to show their robustness with respect to disturbances and uncertainties. This paper focuses on the robustness analysis of linear vibrational control systems with additive disturbances. By applying perturbation techniques, the linear vibrational control system is shown to be input-to-state stable with respect to disturbances. When disturbances are periodic, frequency analysis technique obtains a less conservative estimate of the ultimate bound of the system, indicating that disturbances with high frequencies lead to relatively small ultimate bounds. When additive state-dependent disturbances are considered, weak averaging techniques can be used to show the robustness of the system when bounded disturbances are slow time-varying. Numerical results support the theoretic analysis.

1. Introduction

The vibrational control method was proposed to stabilize systems such as inverted pendulums in an open-loop fashion by inserting a high frequency dither instead of using feedback (Meerkov, 1980). It has been shown that high-frequency dithers can introduce extra design freedom in stabilization and performance improvement (Meerkov, 1980), making it attractive to many engineering applications, see, for example, chemical reactors (Cinar, Deng, Meerkov, & Shu, 1987), gas lasers (Meerkov & Shapiro, 1976) and under-actuated robotics (Hong, 2002; Yabuno, Matsuda, & Aoshima, 2005) and references therein.

We adapt the example of vibrational control system (Khalil, 1996 Example 8.10) to illustrate this idea. By vertically oscillating the suspension point using a sine wave dither with a small amplitude but high frequency (Kapitsa, 1951; Stephenson, 1908), an inverted pendulum can be locally stabilized.[1] The dynamics model of this system is presented as:

$$ml\ddot{\theta} + (mg - ma\omega\cos\omega t + ka\sin\omega t)\sin\theta + kl\dot{\theta} = 0, \tag{1}$$

where θ is the angular displacement, m is the mass, l is the length of pendulum, k is the viscous friction coefficient, a and ω are the amplitude and frequency of oscillating dither respectively. By linearizing it around its equilibrium position at $(\pi, 0)$, the linearized model in state-space becomes:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ \frac{g}{l} & -\frac{k}{m}\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \underbrace{\begin{bmatrix}0 & 0\\ \frac{a\omega}{l}\cos\omega t + \frac{ka}{ml}\sin\omega t & 0\end{bmatrix}}_{B(\omega t)}\begin{bmatrix}x_1\\ x_2\end{bmatrix}. \tag{2}$$

It consists of two parts. One is the unstable A matrix and the other is the periodic matrix $B(\omega t)$ coming from the sinusoidal dither. It has been proved that if the dither frequency ω is sufficiently large, the inverted pendulum is locally stable (Bogoliubov & Mitropolski, 1961). This example shows that even though the equilibrium is unstable, an open-loop controller using a high frequency dither can locally stabilize the system. In this example, though dither is inserted without using feedback, the system (2) has a "feedback-like" structure. This feedback-like behavior in vibrational control design was described as a natural interaction between the system dynamics and vibrated component as pointed out in Shapiro and Zinn (1997).

A thorough analysis of linear vibrational control systems in the form of $\dot{x} = Ax + B(\omega t)x$ was introduced by Meerkov (1980). In his seminal work, it was assumed that A has a controllable canonical form. Under such a scenario, a necessary and sufficient condition to ensure stabilization is that the trace of A is negative.

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[1] The amplitude of dither is selected as $\frac{a}{\omega}$ to simplify the presentation.

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Moreau and Aeyels (2004) used the idea of vibrational control to enlarge the domain of attraction by designing a periodic output feedback for linear-time-invariant (LTI) systems. Similarly, pole assignment capabilities of vibrational control method were discussed in Kabamba, Meerkov, and Poh (1998). Recently, Berg proposed a design tool using the concept of stability maps for a class of second-order linear periodic systems (Berg & Wickramasinghe, 2015).

Subsequently, the framework of nonlinear vibrational control systems was established by R. Bellman and J. Bentsman in Bellman, Bentsman, and Meerkov (1985, 1986a, b). The criteria of stabilization, controllability and transient behavior for different types of vibrational control systems were addressed. Shapiro and Zinn (1997) showed that a class of dynamic system can be locally stabilized by a nonlinear vibrational controller even if its Jacobian matrix has a positive trace. This result shows that the vibrational control method can be applied to a large class of engineering systems.

Although various stability results of vibrational control systems have been published, there is little work addressing the robustness with respect to disturbances or uncertainties, which is one of the most important performance requirements for engineering applications. This work focuses on linear vibrational control systems and explores its robustness in the presence of two types additive disturbances: one is state-independent and the other is state-dependent.

In the motivating example, there are always external forces/moments that can perturb the inverted pendulum. This leads to a linear vibrational control system in the presence of state-independent disturbances:

$$\dot{x} = Ax + B_1(\omega t)x + B_2 w(t),\ x(t_0) = x_0 \in \mathbb{R}^n, \tag{3}$$

where $A \in \mathbb{R}^{n\times n}, B_1 : [t_0, \infty) \to \mathbb{R}^{n\times n}, B_2 \in \mathbb{R}^{n\times m}, w : [t_0, \infty) \to \mathbb{R}^m$. Similarly, state-dependent disturbances such as variation of friction coefficient can also appear thus robustness analysis is important.

One of the key techniques in stability analysis of vibrational control systems is averaging (Bellman et al., 1986b; Shapiro & Zinn, 1997). The existence of disturbances would perturb the averaged systems, leading to undesirable performance. When state-independent disturbances are considered, the perturbation technique (Khalil, 1996) can be applied to show the robustness. When disturbances are bounded and periodic, by using frequency analysis, our result shows that the ultimate bound of the system is inversely proportional to the frequency of the disturbances.

When state-dependent disturbances are considered, neither perturbation method nor frequency analysis can be directly applied. Recently, strong average and weak average techniques have been developed to analyze the robustness of nonlinear time-varying systems when taking disturbances into consideration (Nešić & Teel, 2001). It is shown that the strong average of the vibrational control system does not exist while the weak average exists. By exploring the stability of the weak averaged system, we show that the linear vibrational control system is robust to bounded but slow time-varying disturbances.

The remainder of this paper is organized as follows. In Section 2, preliminaries are stated. Problem formulation and main results are presented in Section 3 for state-independent disturbances. Section 4 discusses the robustness of the linear vibrational control systems with respect to state-dependent disturbances, followed by simulation examples in Section 5. Section 6 concludes the paper. All proofs are provided in Appendix.

2. Preliminaries

The set of real numbers is denoted as $\mathbb{R}$. A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$. It is of class-$\mathcal{K}_\infty$ if it belongs to class-$\mathcal{K}$ and is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if $\beta(\cdot, t)$ belongs to class-$\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is decreasing to zero for each $s > 0$. Define the infinity norm as $\|w\|_\infty := \text{ess sup}_{t\geq 0}|w(t)|$. If $\|w\|_\infty < \infty$, it can be called that $w \in \mathcal{L}_\infty$.

2.1. Vibrational stabilization

In literature, a generic form of vibrational control systems is (Bellman et al., 1985; Bullo, 2002; Meerkov, 1980):

$$\dot{x} = f(x) + g\left(\frac{t}{\varepsilon}, x\right), x(t_0) = x_0 \in \mathbb{R}^n, \forall t \geq t_0 \geq 0, \tag{4}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous with an equilibrium point x_e such that $f(x_e) = 0$. The nonlinear mapping $g : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in both arguments and T-periodic in time.

Definition 1 (*Bellman et al., 1985*). The equilibrium point x_e of $f(x)$ is said to be vibrationally stabilizable (v-stabilizable) if for any $v > 0$ there exists almost periodic and zero-mean $g(\frac{t}{\varepsilon}, x)$ in the first argument such that system (4) has an almost periodic asymptotically stable solution $x^s(t)$ characterized by

$$|\bar{x}^s - x_e| < v,$$

where $\bar{x}^s = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^s(\tau)d\tau$.

As a special case, linear multiplicative vibrational systems in Bellman et al. (1985) characterize a class of linear systems stabilized by a linear vibrational control input:

$$\dot{x} = Ax + \frac{1}{\varepsilon}B_1\left(\frac{t}{\varepsilon}\right)x,\ x(t_0) = x_0 \in \mathbb{R}^n, \tag{5}$$

where the matrix $A \in \mathbb{R}^{n\times n}$ and the vibrational matrix $B_1 : [t_0, \infty) \to \mathbb{R}^{n\times n}$ is continuous and periodic in t with zero mean value (see inverted pendulum (2) for example). The positive number ε serves as the design parameter that is related to the dither frequency.

Lemma 1 provides a necessary and sufficient condition of vibrational stabilization for linear multiplicative systems.

Lemma 1 (*Meerkov, 1980*). *Suppose the matrix A in system (5) has a controllable canonical form, then the system is v-stabilizable if and only if the trace of the matrix A is negative.*

Remark 1. As the trace of a square matrix equals the summation of all its eigenvalues, Lemma 1 implies that there may exist positive eigenvalues such that the system (5) is open-loop unstable . By introducing a high frequency vibration, it is possible to shift unstable eigenvalues to stable ones. ∘

2.2. Input-to-state stability

While there exist disturbances in a dynamic system, input-to-state stability (ISS) (Khalil, 1996) is used to address the robustness. We consider the following time-varying system:

$$\dot{x} = f(t, x, w),\ x(t_0) = x_0 \in \mathbb{R}^n, \forall t \geq t_0 \geq 0, \tag{6}$$

where $f : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous differentiable in t and locally Lipschitz in x and w. The disturbance $w : [t_0, \infty) \to \mathbb{R}^m$ is time-varying. Without losing generality, let us assume $f(t, 0, 0) = 0$.