Automatica 69 (2016) 289–297

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Distributed continuous-time approximate projection protocols for shortest distance optimization problems ☆

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ARTICLE INFO

Article history:
Received 6 April 2014
Received in revised form
2 February 2016
Accepted 4 February 2016

Keywords:
Distributed optimization
Convex intersection
Shortest distance optimization
Approximate projection

ABSTRACT

In this paper, we investigate a distributed shortest distance optimization problem for a multi-agent network to cooperatively minimize the sum of the quadratic distances from some convex sets, where each set is only associated with one agent. To deal with this optimization problem with projection uncertainties, we propose a distributed continuous-time dynamical protocol, where each agent can only obtain an approximate projection and communicate with its neighbors over a time-varying communication graph. First, we show that no matter how large the approximate angle is, system states are always bounded for any initial condition, and uniformly bounded with respect to all initial conditions if the inferior limit of the stepsize is greater than zero. Then, in both cases of nonempty and empty intersection of convex sets, we provide stepsize and approximate angle conditions to ensure the optimal convergence, respectively. Moreover, we also give some characterizations about the optimal solutions for the empty intersection case.

1. Introduction

In recent years, distributed optimization of a sum of convex functions has attracted much attention due to its wide applications in resource allocation, source localization and robust estimation (referring to Bertsekas & Tsitsiklis, 1989, Jakovetic, Xavier, & Moura, 2011, Johansson, Rabi, & Johansson, 2009, Lu & Tang, 2012, Lu, Tang, Regier, & Bow, 2011, Nedić & Ozdaglar, 2008, Nedić & Ozdaglar, 2009 and Nedić, Ozdaglar, & Parrilo, 2010). A whole optimization task can be accomplished cooperatively by a group of autonomous agents via simple local information exchange and distributed protocol design even when the communication graph among agents is time-varying.

Although many existing distributed optimization works have been done by discrete-time algorithms, more and more attention has been paid to continuous-time algorithms in recent years (Droge, Kawashima, & Egerstedt, 2014; Gharesifard & Cortés, 2014; Kvaternik & Pavel, 2012; Shi, Johansson, & Hong, 2013; Wang & Elia, 2010, 2011), partially because continuous-time models can be studied by various well-developed continuous-time methods or make the algorithms easily implemented in physical systems. A distributed continuous-time computation model was proposed to solve an optimization problem for a fixed undirected graph in Wang and Elia (2010), with the optimization achieved by controlling the sum of subgradients of convex functions, and later this model was generalized to weight balanced graph case in Gharesifard and Cortés (2013) for differentiable objective functions with globally Lipschitz continuous gradient. Another continuous-time distributed algorithm with a constant stepsize was developed in Kvaternik and Pavel (2012) for optimization problems with positivity constraints in a fixed undirected graph case, where a lower bound of convergence rate and an upper bound on the agents' estimate error were presented. Moreover, the relationship between the existing dual decomposition and consensus-based methods for distributed optimization was revealed in Droge et al. (2014), where both approaches were based on the subgradient method, but one with a proportional control term and the other with an integral control term.

When the optimal solution sets of agents' individual convex objective functions have a nonempty intersection, the distributed optimization problem is equivalent to convex intersection problems

☆ This work was supported by the National Natural Science Foundation of China under Grant 71401163 and 61333001, Beijing Natural Science Foundation under Grant 4152057, Hong Kong Research Grants Council under Grant 419511 and Hong Kong Scholars Program under Grant XJ2015049. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Claudio de Persis under the direction of Editor Christos G. Cassandras.

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http://dx.doi.org/10.1016/j.automatica.2016.02.019
0005-1098/

(CIP) (Deutsch, 1983; Gubin, Polyak, & Raik, 1967; Lou, Shi, Johansson, & Hong, 2013, 2014; Nedić et al., 2010; Shi & Johansson, 2012; Shi et al., 2013). A projected consensus algorithm was proposed in Nedić et al. (2010) for a network to solve CIP, where all agents were shown to converge to a common point in the intersection set for weight-balanced and jointly connected communication graphs. Later, a continuous-time dynamical system was designed and various connectivity conditions were discussed in Shi et al. (2013). In addition, a random sleep algorithm was proposed with providing conditions to converge almost surely to a common point in the intersection set in Lou et al. (2013), with agents to randomly take the neighbor-based average or projection onto their individual sets based on a Bernoulli process. Almost all the existing optimization results were obtained based on the assumption that the exact projection point onto convex sets can be obtained (Deutsch, 1983; Gubin et al., 1967; Lin & Ren, 2012; Meng, Xiao, & Xie, 2013; Nedić et al., 2010; Shi & Johansson, 2012; Shi et al., 2013).

On the other hand, the intersection of convex optimal solution sets may be empty in practice. In this case, how to seek a point with the shortest (quadratic) distance to these sets is also important. For instance, the supply center location problem is concerned with how to seek the location of raw materials supply center so that the average transportation cost from this supply center to the multiple factories is minimal (Francis, McGinnis, & White, 1992; Pardalos & Romeijn, 2002); the source localization in a sensor network is related to estimate the location of the source emitting a signal based on the received signals of multiple sensors in a noisy environment (Meng et al., 2013; Zhang, Lou, Hong, & Xie, 2015). In fact, the problem for both empty and nonempty intersection cases is referred to as the shortest distance optimization problem (SDOP). Clearly, CIP is a special case of SDOP, and the average consensus problem is also a special case of SDOP since the optimal solution of the minimum of the sum of quadratic functions from some points is exactly the average of these points. Some distributed algorithms were proposed to discuss SDOP. For example, Meng et al. (2013) formulated a source localization problem as the SDOP in a plane and proposed a discrete-time distributed algorithm, with the adjacency matrices of communication graphs required to be doubly stochastic. Moreover, Lin and Ren (2012) proposed two distributed continuous-time algorithms to solve SDOP in the empty intersection case for connected graphs: the first one was designed for optimal consensus based on sign functions, and the second one was modified to avoid chattering but only to achieve the optimal consensus approximately.

The objective of this paper is to design a continuous-time distributed protocol to solve SDOP based on approximate projection. Note that the exact projection is usually hard to obtain in practice. Therefore, approximate projection issues were discussed in some situations, and for example, Lou et al. (2014) proposed a discrete-time approximate projected consensus algorithm to solve CIP. The motivation of the current research aims at cooperatively solving SDOP with projection uncertainties and continuous-time dynamics. For example, in a practical robotic network to solve the SDOP, a robot may not always obtain the exact projection point of its own convex set, but only spot some point on the set surface near the exact projection point. The contribution of this paper can be summarized as follows.

- We propose a new concept of approximate projection when the exact projection is hard to obtain. In fact, we consider an approximate projection related to set boundary surfaces, different from that defined in a "triangle" in Lou et al. (2014). To overcome the difficulties resulting from this new approximate projection, we employ a geometric method to convert the original problem into a heterogeneous stepsize problem.
- Given any approximate angle, we show that, with the proposed continuous-time algorithm, the agent states are always bounded for any initial condition, and uniformly bounded with respect to all initial conditions if the stepsize is not too small. The result with respect to the continuous-time algorithm is different from some results based on some discrete-time ones. In fact, $\pi/4$ was shown to be the critical approximate angle for the boundedness of the discrete-time algorithm with the approximate projection defined in a triangle in Lou et al. (2014).
- We study SDOP in both nonempty and empty intersection cases, and propose a unified protocol based on the approximate projection. In fact, the proposed convergence conditions and proofs in the two cases are quite different. Note that our result is different from that in Lin and Ren (2012) because we handle approximate projections without assuming that the communication graph is always connected, and ours tackles both nonempty and empty intersection cases, while Lou et al. (2014) only does the nonempty intersection case.

Notations: $\mathbf{1}$ denotes the vector with all ones; y^T denotes the transpose of a vector $y \in \mathbb{R}^m$; $|y|$ denotes the Euclidean norm of y; $[v, z]$ denotes the line segment connecting the two points v, z; line(v, z) denotes the line passing the two points v, z; for a set $K \subseteq \mathbb{R}^m$, int(K) and $bd(K) = K\backslash\text{int}(K)$ denote the sets of interior points and boundary points of K, respectively; $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product in $\mathbb{R}^m$; the angle between nonzero vectors y and z is denoted as $\angle(y, z) \in [0, \pi]$, where $\cos\angle(y, z) = \langle y, z\rangle/(|y|\,|z|)$; span$\{v_1, \ldots, v_p\}$ (aff$\{v_1, \ldots, v_p\}$) denotes the subspace (affine hull) generated by vectors $v_1, \ldots, v_p$.

2. Preliminaries

2.1. Graph theory

A multi-agent network can be described by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the node (or agent) set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the arc set with the arc $(j, i) \in \mathcal{E}$ describing that node i can receive the information of node j. Here $(i, i) \notin \mathcal{E}$ for all i. Let $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$ be the set of neighbors of node i. A path from node i to node j in $\mathcal{G}$ is a sequence $(i, i_1), (i_1, i_2), \ldots, (i_p, j)$ of arcs in $\mathcal{E}$. Graph $\mathcal{G}$ is said to be strongly connected if there exists a path from i to j for each pair of nodes $i, j \in \mathcal{V}$. Graph $\mathcal{G}$ is undirected when $(j, i) \in \mathcal{E}$ if and only if $(i, j) \in \mathcal{E}$ (Godsil & Royle, 2001).

The communication over the network under consideration is switching and characterized by a directed graph process $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$, $t \geq 0$, with $\mathcal{E}_{\sigma(t)}$ the arc set of the graph at time t. Here $\sigma : [0, \infty) \to \mathcal{Q}$ is a piecewise constant function to describe the time-varying graph process, where $\mathcal{Q}$ is the index set of all possible graphs on $\mathcal{V}$. Let $\Delta := \{t_k, k \geq 0\}$ with $t_0 = 0$ denote the set of all switching moments of switching graph $\mathcal{G}_\sigma$. As usual, we assume there is a dwell time $\tau > 0$ between two consecutive graph switching moments, i.e., $t_{k+1} - t_k \geq \tau$ for all k. The switching graph $\mathcal{G}_\sigma$ is uniformly jointly strongly connected (UJSC) if there exists $T > 0$ such that the union graph $(\mathcal{V}, \cup_{t\leq s<t+T} \mathcal{E}(s))$ is strongly connected for $t \geq 0$.

2.2. Convex analysis

A set $K \subseteq \mathbb{R}^m$ is convex if $\lambda z_1 + (1 - \lambda)z_2 \in K$ for any $z_1, z_2 \in K$ and $0 < \lambda < 1$. For a closed convex set K in $\mathbb{R}^m$, we can associate with any $z \in \mathbb{R}^m$ a unique element $P_K(z) \in K$ satisfying $|z - P_K(z)| = \inf_{y\in K} |z - y| =: |z|_K$, where P_K is called the projection operator onto K. We have the following properties for the projection operator P_K (Rockafellar, 1972).