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Stability analysis and robust control of heart beat rate during treadmill exercise☆

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ABSTRACT

We investigate a nonlinear dynamical model of a human's heart beat rate (HBR) during a treadmill exercise. We begin with a rigorous analysis of the stability of the model that extends significantly the results available in the literature. In particular, we first identify a simple set of necessary and sufficient conditions for both input-state stability and Lyapunov stability of the system, and then prove that the same conditions also hold when the model parameters are subject to unknown but bounded perturbations. The second part of the paper is devoted to the design and analysis of a control structure for this model, where the treadmill speed plays the role of the control input and the output is the subject's HBR, which is intended to follow a prescribed pattern. We propose a simple control scheme, suitable for a practical implementation, and then analyze its performance. Specifically, we prove (i) that the same conditions that guarantee the stability of the system also ensure that the controller attains a desired level of performance (quantified in terms of the admissible deviation of the HBR from the prescribed profile) and (ii) that the controller is robust to bounded perturbations both in the system parameters and the control input. Numerical simulations are also presented in order to illustrate some of the theoretical results.

1. Introduction

A subject's heart beat rate (HBR) can vary as the body's need to absorb oxygen and excrete carbon dioxide changes, such as during sleep, illness and, in particular, physical exercise. Because each individual has a constant blood volume, one of the physiological ways to deliver more oxygen to an organ is to increase the HBR to make blood pass through the organ more often. These well known biological facts, together with the availability of inexpensive measurement equipment, have made HBR a widely used indicator of exercise intensity.

Indeed, HBR monitoring helps physicians manage and control exercise training intensity in order to ensure the subject's safety during the practice. It is also common for physicians to individualize HBR profiles by taking into account the physiological state of the subject. Such practice clearly calls for appropriate i.e., accurate, flexible and reliable models of the HBR. One of the most promising such models was introduced in Cheng, Savkin, Celler, Su, and Wang (2008), where a specific nonlinear input–output relationship linking the HBR (output) and the treadmill speed (input) is described. The model in Cheng, Savkin and Celler et al. (2008) has two state variables. One represents the deviation of the HBR from the subject's beat rate when at rest, while the second variables model internal peripheral effects.

The contributions in Cheng, Savkin and Celler et al. (2008) include a controller to regulate the HBR, given a suitably defined input signal related to an a priori prescribed HBR profile. This control scheme, however, is based on a linear approximation of the original *nonlinear* model. Moreover, the stability of the system cannot be guaranteed in the presence of perturbations, either to the model parameters or to the input signal. Additional attempts to design controllers based on the model of Cheng, Savkin and Celler et al. (2008) can be found in Cheng, Savkin, Su, Celler, and Wang (2008), Mazenc, Malisoff, and De Querioz (2011) and Patrascu, Patrascu, and Hantiu (2014). However, only Mazenc et al. (2011)

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provides a control schemes based on the original nonlinear model of Cheng, Savkin and Celler et al. (2008), rather than a linear approximation.

In Mazenc et al. (2011) it is argued that, since the controller only involves one out of five different parameters in the model, the scheme is robust to parameter perturbations. However, the sufficient conditions for the validity of the control method which are derived in Mazenc et al. (2011) involve the HBR profile and the exact parameter values. Unfortunately, parameter mismatches, even if slight, easily prevent these conditions from being satisfied (as we specifically show in Section 4; see Remark 3). The computer simulations in Mazenc et al. (2011) actually involve a very specific choice of the parameter values (which guarantee stability).

While robust controllers for the HBR model of interest have been proposed in the original paper Cheng, Savkin and Celler et al. (2008) and its sequel Cheng, Savkin and Su et al. (2008), they are based on the H_∞ method and, therefore, rely on a linearization of the original model. The scheme in Cheng, Savkin and Su et al. (2008), in particular, is designed to cope with additive noise in the linearized system, however the model parameters are assumed to be known exactly. To summarize, none of the control methods proposed in Cheng, Savkin and Celler et al. (2008); Cheng, Savkin and Su et al. (2008) and Mazenc et al. (2011) is designed to account for perturbations to the input signal and they require the model parameters to be known either exactly (Cheng, Savkin and Celler et al., 2008; Cheng, Savkin and Su et al., 2008) or with great accuracy (Mazenc et al., 2011). The control scheme in Patrascu et al. (2014) relies on a linear approximation of the system and it is only evaluated through computer simulations, not analytically.

The contribution of this paper is twofold. We start with a rigorous stability analysis of the nonlinear model of Cheng, Savkin and Celler et al. (2008). The study of the system stability carried out in Cheng, Savkin and Celler et al. (2008) revolves around the implications of a certain bilinear matrix inequality (BMI) and it is only valid for *exact* values of the parameters. By converting the BMI into a linear matrix inequality (LMI), we extend the stability analysis to handle *intervals* of values of the model parameters. To be precise, we find necessary and sufficient conditions, which are satisfied within a range of parameter values, for the model to be Lyapunov stable and input-state stable (Lyapunov, 1966; Sontag & Wang, 1995). We also prove that the same stability conditions still hold when the model parameters are subject to unknown but bounded perturbations, either deterministic or random.

The second contribution of the paper is the design and analysis of a control scheme that adjusts the input signal (i.e., the treadmill speed) to make the subject's HBR follow the rate profile prescribed by a physician. The structure of the controller is simple enough for practical implementations. We quantify the controller performance in terms of the admissible deviation of the HBR from the prescribed profile and prove that the controlled system is stable as long as the sufficient conditions for the stability of the original model are satisfied.[1] This implies, in particular, that the control scheme is robust to bounded perturbations of the model parameters and its implementation demands only that the bounds for the variation of a single parameter are determined a priori. Our approach overcomes the main limitations in Cheng, Savkin and Celler et al. (2008); Cheng, Savkin and Su et al. (2008) and Mazenc et al. (2011), namely the need to have an accurate knowledge of the model parameters and the need to have a non-perturbed input signal. It also enables the analysis of the original *nonlinear* system in Cheng, Savkin and Celler et al. (2008) without resorting to linearizations or other approximations.

[1] The necessary and sufficient conditions for stability are nearly identical. They are given in terms of two inequalities that differ at a single point.

Finally, we provide a set of computer simulations to illustrate the theoretical results and show the effectiveness of the proposed method. In order to numerically assess the robustness of the technique, the model parameters in the simulations are not held constant but they evolve over time within the 95% confidence interval provided in Cheng, Savkin and Celler et al. (2008). The input signal is also distorted in the simulations by applying a multiplicative perturbation.

The rest of the paper is organized as follows. Section 2 contains the model description. Section 3 is devoted to the stability analysis. In Section 4 we introduce the control scheme and analyze its performance, including the case of uncertain parameters and perturbed input signal. Numerical simulations are presented in Section 5 and Section 6 is devoted to a concluding discussion.

2. Model description

The model proposed in Cheng, Savkin and Celler et al. (2008) to simulate the HBR is a system of two differential equations,

$$\begin{aligned} \dot{x}_1(t) &= -a_1 x_1(t) + a_2 x_2(t) + a_2 u^2(t) \\ \dot{x}_2(t) &= -a_3 x_2(t) + \phi(x_1(t)), \end{aligned} \tag{1}$$

where ϕ is a nonlinearity defined as

$$\phi(y) = \frac{a_4 y}{1 + \exp(-(y - a_5))},$$

$a_i > 0$, $i = 1, \ldots, 5$, are positive and static model parameters, $x_1(t)$ is proportional to the deviation of the instantaneous HBR from the nominal rate when the subject is at rest (in particular, the instantaneous HBR is $h(t) = 4x_1(t) + 74$) and $x_2(t)$ represents the superposition of various internal, and typically slower, processes that take place in the body during the exercise and affect the HBR. Changes in types and density of hormones, boosted metabolism and the increase of body temperature are some examples of such processes (see Cheng, Savkin and Celler et al. (2008) for additional details and examples). The input signal $u(t)$ is the treadmill speed, which serves as an indicator of the exercise intensity. Note that, according to Eq. (1) and the definition of ϕ, the signals $x_1(t)$, $x_2(t)$ and $u(t)$ are always positive (Cheng, Savkin and Celler et al., 2008). In the rest of the paper, we denote $v(t) = a_2 u^2(t)$ for simplicity.

Model (1) was calibrated in Cheng, Savkin and Celler et al. (2008) using several data sets and the parameters a_i, $i = 1, \ldots, 5$, were estimated to be contained in 95% confidence intervals of the form $a_i \in (\hat{a}_i - \delta_i, \hat{a}_i + \delta_i)$ where

$$\begin{aligned} &\hat{a}_1 = 1.84, \quad \hat{a}_2 = 24.32, \quad \hat{a}_3 = 0.0636, \\ &\hat{a}_4 = 0.00321, \quad \hat{a}_5 = 8.32, \end{aligned} \tag{2}$$

and

$$\begin{aligned} &\delta_1 = 0.36, \quad \delta_2 = 4.36, \quad \delta_3 = 1.95 \times 10^{-2}, \\ &\delta_4 = 6.84 \times 10^{-4}, \quad \delta_5 = 0.44. \end{aligned} \tag{3}$$

3. Stability analysis

It was shown in Cheng, Savkin and Celler et al. (2008) that there exists a positive definite matrix P such that the matrix inequality

$$A_1^\top P + P A_1 + P B_1 B_1^\top P + C^\top C < 0 \tag{4}$$

is satisfied with

$$A_1 = \begin{pmatrix} -\hat{a}_1 & \hat{a}_2 \\ 0 & -\hat{a}_3 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 \\ \hat{a}_4 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \tag{5}$$