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Brief paper

# Spectral tests for observability and detectability of periodic Markov jump systems with nonhomogeneous Markov chain[☆]

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ABSTRACT

This paper addresses observability and detectability of discrete-time periodic systems with nonhomogeneous Markov jump parameter. Popov–Belevitch–Hautus (PBH)-type criteria are proposed for the concerned structural properties. It is shown that, different from stochastic systems with constant coefficients or homogeneous Markov chain, the spectral criteria of considered plants do not rely on a single operator but on a finite sequence of linear evolution operators. By use of the obtained detectability criterion, an extended Lyapunov theorem is established, which relates asymptotic mean square stability to a periodic Lyapunov equation. Further, a difference Riccati equation with periodic coefficients is studied and some sufficient conditions are presented for the existence of stabilizing solution.


## 1. Introduction

Observability and detectability are two fundamental concepts in the modern control theory. It is well known that, for linear constant systems, spectral criteria are available to examine these properties in terms of the eigenvalues of "$A$" matrix, which are the famous PBH tests (see Rugh, 1996). Due to widespread applications, stochastic systems have received considerable attention in recent years (Costa, Fragoso, & Todorov, 2012; Petersen, Ugrinovskii, & Savkin, 2000). By means of different formulations, several structural concepts have been introduced for various stochastic models, such as exact controllability (Liu & Peng, 2002), null controllability (Shen, Sun, & Wu, 2013a), exact observability (Zhang & Chen, 2004) and exact detectability (Zhang, Zhang, & Chen, 2008) of stochastic Itô systems, W-observability/detectability of Markov jump systems (Costa & Do Val, 2001, 2002), stochastic observability/detectability of Markov jump systems with multiplicative noise (Dragan & Morozan, 2006; Dragan, Morozan, & Stoica, 2010). Particularly, in parallel with the deterministic PBH tests, stochastic PBH tests have been established based on the spectrum of associated Lyapunov operators. More specifically, stochastic PBH criterion was first developed by Zhang and Chen (2004) for observability of linear stochastic Itô systems. Later, exact detectability was proposed in Zhang et al. (2008), where the spectral criterion of detectability was supplied to settle an infinite horizon $H_2/H_\infty$ control problem. As the latest progress on this issue, Ni, Zhang, and Fang (2010) and Shen, Sun, and Wu (2013b) have generalized the works of Zhang and Chen (2004), Zhang et al. (2008) to continuous- and discrete-time linear Markov jump systems with multiplicative noises, respectively. More importantly, it is clarified that the notions of W-detectability (resp. stochastic observability) and exact detectability (resp. exact observability) are really equivalent (Shen et al., 2013b), which exhibits the effectiveness of spectral criteria. Despite the great achievements in structural analysis of stochastic systems, it should be pointed out that the existing stochastic PBH criteria are all devoted to stochastic systems with time-invariant coefficients or homogeneous Markov chain. This is mainly because the current spectrum analytic technique relies severely on a Lyapunov operator arising from the second-order state moment of related stochastic systems. If the system coefficients or the transition probability of Markov chain are time-varying, the spectrum set of Lyapunov operator will be no more stationary. Hence, the previous spectral approach fails to treat stochastic systems with time-varying coefficients or nonhomogeneous Markov process.

In this paper, we aim to set up PBH-type criteria for observability and detectability of discrete-time periodic Markov jump systems (DPMJS). The system coefficients and transition probability

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matrix of Markov jump parameter allow to be periodically time-varying. Such type of models have been applied to communication networks (Aberksne & Dragan, 2012), portfolio optimization (Costa & Araujo, 2008), etc. Moreover, control issues of Markov jump systems with time-varying coefficients or nonhomogeneous Markov process have been extensively researched; see Aberksne (2011), Dragan et al. (2010), Dragan, Morozan, and Stoica (2014), Ma and Jia (2013), Ma, Zhang, and Hou (2012) and references therein. However, to the best of our knowledge, there is no similar result reported for the structural properties of such dynamics up to now. To fill this gap, this work is to develop a spectrum analytic approach suitable for analyzing DPMJS with periodically time-varying transition probability. The key lies in a novel theoretical tool, named "monodromy operator", which is first put forward and will play a central role in the structural analysis of DPMJS. By means of the monodromy operator, spectral criteria are successfully established for observability and detectability of DPMJS. In contrast to the stochastic PBH criteria for stochastic time-invariant systems (Qi, 2008; Zhang & Chen, 2004; Zhang et al., 2008), or homogeneous Markov jump systems (Ni et al., 2010; Shen et al., 2013b), it is demonstrated that the spectral criteria of DPMJS not only depend on the monodromy operator, but also have close connections with a sequence of linear evolution operators. As theoretical application, we will employ the obtained spectral criteria to study the asymptotic mean square stability of DPMJS, for which an extended Lyapunov theorem is presented in terms of a periodic Lyapunov equation. Besides, a class of difference Riccati equations with periodic coefficients are studied, and some sufficient conditions for the existence of stabilizing solution are supplied under the assumptions of observability and detectability.

The rest of this paper is organized as follows. In Section 2, we introduce a monodromy operator, and then give a spectral characterization of stability. Based on the spectrum of the monodromy operator, PBH-type criteria of observability and detectability with their applications are presented in Section 3. Finally, Section 4 ends this paper with a concluding remark.

**Notations.** $\mathcal{C}^n(R^n)$: $n$-dimensional complex (real) space with Euclidean norm $\|\cdot\|$; $R^{n\times m}$: the space of $n\times m$ real matrices with operator norm $\|\cdot\|_2$; $S_n$: the set of $n\times n$ symmetric matrices, whose entries may be complex; $A>0(\geq 0)$: $A$ is positive (semi-)definite; $A'$: the transpose of $A$; $I_n$: the $n\times n$ identity matrix; $\mathbb{Z}_+=\{0,1,\ldots\}$ and $\mathbb{Z}_{1+}=\mathbb{Z}_+/\{0\}$; $\mathcal{C}$: the set of complex numbers; $\otimes$: the operation of Kronecker product; Ker$(\cdot)$: the kernel of a matrix; diag$\{\cdot\}$: a (block-)diagonal matrix; $\mathbf{1}(\cdot)$: the indicator function.

## 2. Preliminaries

On a complete probability space $(\Omega,\mathscr{F},\mathscr{P})$, we consider the following discrete-time linear periodic Markov jump systems:

$$\begin{cases} x(t+1)=A(t,\eta_t)x(t),\\ z(t)=C(t,\eta_t)x(t), \quad t\in\mathbb{Z}_+,\end{cases} \tag{1}$$

where $x(t)\in R^n$ and $z(t)\in R^{n_z}$ denote the system state and measurement output, respectively. Let $\{\eta_t\}_{t\in\mathbb{Z}_+}$ be a Markov chain with the state space $\mathcal{D}$ and the transition probability matrix $\mathcal{P}_t=[p_t(i,j)]_{N\times N}$. There hold $p_t(i,j)\geq 0$ and $\sum_{j=1}^N p_t(i,j)=1$ for $t\in\mathbb{Z}_+$ and $i\in\mathcal{D}=\{1,2,\ldots,N\}$. The initial distribution of $\eta_t$ at $t_0$ is represented by $\pi_{t_0}=(\pi_{t_0}(1),\ldots,\pi_{t_0}(N))$ where $\pi_{t_0}(i):=\mathcal{P}(\eta_{t_0}=i)\geq 0$. Moreover, all the coefficients of (1) are $\theta$-periodic matrices of suitable dimensions, e.g., $A(t,i)=A(t+\theta,i)\in R^{n\times n}$, and the transition probability of $\eta_t$ satisfies $p_t(i,j)=p_{t+\theta}(i,j)$ $(i,j\in\mathcal{D})$, where $\theta\in\mathbb{Z}_{1+}$.

The following definitions are basic in the subsequent analysis.

**Definition 1** (*Dragan et al., 2010*)**.** The zero state equilibrium of autonomous discrete-time periodic Markov jump systems:

$$x(t+1)=A(t,\eta_t)x(t), \quad t\in\mathbb{Z}_+ \tag{2}$$

or $(\mathbb{A};\mathbb{P})$ is asymptotically mean square stable (AMSS) if $\lim_{t\to\infty}E\|x(t;x(t_0),\pi_{t_0})\|^2=0$ for any $t_0\in\mathbb{Z}_+$, $x(t_0)\in R^n$, and arbitrary initial distribution $\pi_{t_0}$ of the Markov chain.

**Definition 2.** System (1) or $(\mathbb{A},\mathbb{C};\mathbb{P})$ is observable at time $t_0$ if for arbitrary initial distribution $\pi_{t_0}$ of the Markov chain, there holds

$$z(t)\equiv 0 \quad \text{(a.s.) for } t\in[t_0,\infty)\ \Rightarrow x(t_0)=0 \text{ (a.s.)}. \tag{3}$$

Moreover, $(\mathbb{A},\mathbb{C};\mathbb{P})$ is (uniformly) observable if it is observable for all $t_0\in\mathbb{Z}_+$.

**Definition 3.** $(\mathbb{A},\mathbb{C};\mathbb{P})$ is (uniformly) detectable if for any $t_0\in\mathbb{Z}_+$, $x(t_0)\in R^n$, and arbitrary initial distribution $\pi_{t_0}$ of the Markov chain, there holds

$$z(t)\equiv 0 \quad \text{(a.s.) for } t\in[t_0,\infty)$$
$$\Rightarrow \lim_{t\to\infty}E\|x(t;x(t_0),\pi_{t_0})\|^2=0. \tag{4}$$

**Remark 1.** As usual, if $x(t_0)=\xi\in R^n$ is a state such that $z(t)\equiv 0$ almost surely for all $t\geq t_0$, then $\xi$ is called an unobservable state of (1) at $t_0$. It can be checked that the set of all unobservable states of (1) at each $t\in\mathbb{Z}_+$, denoted by $\bar{\mathcal{O}}_t$, is a subspace of $R^n$. Obviously, $(\mathbb{A},\mathbb{C};\mathbb{P})$ is observable at $t$ if and only if (iff) $\bar{\mathcal{O}}_t=\{0\}$.

Let $S_n^N$ (resp. $S_n^{N+}$) be the set of all $N$ sequences $V=(V(1),\ldots,V(N))$ with $V(i)\in S_n$ (resp. $V(i)\geq 0$). Thus, $S_n^N$ is a Hilbert space with the inner product:

$$\langle U,V\rangle=\sum_{i=1}^N \mathrm{Tr}(U(i)V(i)), \quad \forall U,V\in S_n^N. \tag{5}$$

Define a Lyapunov operator $\mathscr{L}_t:S_n^N\to S_n^N$ as $\mathscr{L}_t(U)=(\mathscr{L}_t(U,1),\ldots,\mathscr{L}_t(U,N))$, where

$$\mathscr{L}_t(U,i)=\sum_{j=1}^N p_t(j,i)A(t,j)U(j)A(t,j)', \quad \forall U\in S_n^N. \tag{6}$$

Associated with the inner product (5), the adjoint operator of $\mathscr{L}_t$ is given by $\mathscr{L}_t^*(U)=(\mathscr{L}_t^*(U,1),\ldots,\mathscr{L}_t^*(U,N))$:

$$\mathscr{L}_t^*(U,i)=A(t,i)'\sum_{j=1}^N p_t(i,j)U(j)A(t,i), \quad \forall U\in S_n^N. \tag{7}$$

Based on $\mathscr{L}_t$, we can get a causal evolution $\mathscr{T}_{t,s}=\mathscr{L}_{t-1}\cdots\mathscr{L}_s$ $(t>s\geq 0)$, which is easily verified to be a linear positive operator. When $t=s$, $\mathscr{T}_{t,t}=\mathscr{I}$ (i.e., the identity operator).

**Definition 4.** $\mathscr{T}_t^\theta=\mathscr{T}_{t+\theta,t}$ is called the monodromy operator of $(\mathbb{A};\mathbb{P})$.

From (6), it is obvious that $\mathscr{L}_{t+\theta}=\mathscr{L}_t$, thus the following properties are straightforward.

**Proposition 1.** *(i)* $\mathscr{T}_{t,s}=\mathscr{T}_{t+\theta,s+\theta}$; *(ii)* $\mathscr{T}_t^\theta=\mathscr{T}_{t+\theta}^\theta$.

By using the $\mathscr{H}$-representation method (Zhang & Chen, 2012), there are a unique triple of constant matrices $H\in R^{n^2N\times\frac{n(n+1)}{2}N}$, $M_t\in R^{n^2N\times n^2N}$ and $L_t\in R^{\frac{n(n+1)}{2}N\times\frac{n(n+1)}{2}N}$ such that

$$\begin{cases}\psi(\mathscr{L}_t(X))=M_t\psi(X), \quad \varphi(\mathscr{L}_t(X))=L_t\varphi(X),\\ \psi(X)=H\varphi(X), \quad \forall X\in S_n^N.\end{cases} \tag{8}$$