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Brief paper

Observability criteria for impulsive control systems with applications to biomedical engineering processes ☆

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ABSTRACT

One of the fundamental properties of the impulsive systems is analyzed: observability. Algebraic criteria for testing this property are obtained for the nonlinear case, considering continuous and discrete outputs. For this class of systems, observability is explored not only through time derivatives of the output, but also considering few discrete measurements at different time-instants. In this context, it is shown that nonlinear impulsive control systems can be strongly observable or observable over a finite time interval. A new rank condition based on the structure of the impulses is found to characterize observability of linear impulsive systems. It generalizes the celebrated Kalman criterion, for both kind of outputs, discrete and continuous. Finally, these results are tested and illustrated both on academic examples and on two impulsive dynamical models from biomedical engineering science.

1. Introduction

Impulsive control systems (ICS) are encountered in various areas as biology, health, robotics and others. For instance, a diabetic type I patient model will be shortly considered herein, for which new specific mathematical tools are needed for analysis, observation and control. Glycemia regulation is performed in real life by appropriate insulin injections and eventually compensatory snacks, to maintain glucose levels within the predefined target range. These inputs can be approximated as impulses whenever insulin bolus is injected, and are adjusted based on discrete glycemia measurements from blood samples taken at various times during the day (Huang, Li, Song, & Guo, 2012). The intake of 'meals' will affect the level of glucose of the patient, and therefore is considered as an impulse disturbance. In this context, ICS seems the appropriate tool to analyze its dynamics.

Another interesting example of ICS is the model of the dynamics of the human immunodeficiency virus (HIV), initially described in Perelson, Kirschner, and Boer (1993). The intake of drugs once or twice a day can be interpreted as an impulse input (Bellman, 1971), with a fixed time interval. Besides, the measurement of its outputs are far from being continuous since blood samples are taken at most every three or six months. In this framework, ICS is a more pragmatic point of view. The accessibility of this ICS was explored in Rivadeneira and Moog (2012).

More generally, impulsive control systems define a class of systems whose state trajectories are piecewise continuous, with discontinuities of the first kind or 'jumps' at some discrete time instants. The dynamics is modeled by algebraic discrete equations or by introducing impulses into the differential equations.

Observability in linear ICS has been investigated in Guan, Qian, and Yu (2002), Medina and Lawrence (2008), Shi and Xie (2012) and Xie and Wang (2005). The definition used therein establishes that observability depends on measurements of the output on a finite-time interval $[0, t_f]$. When a continuous output is considered, the most popular tool remains a Kalman type observability matrix $\mathcal{O}$ (Guan et al., 2002; Xie & Wang, 2005; Zhao & Sun, 2009), but with a very restrictive assumption over the class of considered impulsive systems. Discontinuities in the state of the form $x(\tau_k^+) = A_I x(\tau_k)$ are allowed, where A_I defines a diagonal matrix. A different class of impulsive control systems is considered in Medina and Lawrence (2009), for which the states evolve in continuous form but the output is available for measurement at discrete times only. Suitable criteria based on geometric properties of the invariant observable space and the observability Gramian were worked out for this case.

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In this paper and for the first-time ever, observability is investigated in nonlinear ICS. The dual property of accessibility of nonlinear ICS was characterized in Rivadeneira and Moog (2012) and the basis of the impulsive exact linearization was stated.

The results of the paper are in threefold: (i) The sufficient and necessary conditions are provided to test observability for linear systems with discrete time outputs. This condition has to be viewed as an extended Kalman criterion for observability. Also, the equivalence between the algebraic condition and the observability Gramian is detailed. (ii) The nonlinear case with continuous output and discrete measurements is tackled. Two definitions of observability are introduced with their respective criterion. Strong observability and observability over a finite time interval, are more natural for the latter nonlinear ICS. (iii) Observability is tested on two important models borrowed from biomedical engineering science: HIV and diabetic type I patient models. A brief description of the glycemia dynamical model is given for diabetic patients in the framework of ICS.

2. Preliminaries

A plant is an impulsive control system when there is a set of time instants $T = \{\tau_k\}$, $\tau_k \in \mathbb{R}, \tau_k < \tau_{k+1} < \infty$, and a set of inputs $U_k \in \mathbb{R}^n$, $k = 1, 2, \ldots$, such that the state $x \in \mathbb{R}^n$ is discontinuous at each τ_k according to $x(\tau_k^+) = f_I(x(\tau_k)) + U(k, x)$. Note that the control instants are not necessarily equidistant, the control $U(k, x)$ yields a discontinuity of x at instant τ_k, the function $f_I(x)$ defines discontinuities of the first kind (or 'natural jumps') in the state variable, and the system is left-continuous, *i.e.* $x(\tau_k^-) = x(\tau_k)$.

The class of dynamic systems of interest basically consists of objects defined by a set of impulsive first-order differential equations of the form (Rivadeneira & Moog, 2012; Yang, 2001)

$$\begin{cases} \dot{x}(t) = f(x), & x(t_0) = x(t_0^+) = x_0, \quad t \neq \tau_k, \\ x(\tau_k^+) = f_I(x(\tau_k)) + g(x(\tau_k))u(\tau_k), & t = \tau_k, \ k \in \mathbb{N}, \\ y_c(t) = h_c(x(t)), & \text{or} \\ y_d[k] = h_d(x(\tau_k)), & k \in \mathbb{N} \end{cases} \tag{1}$$

where the state $x \in \mathcal{X} \in \mathbb{R}^n$, the input $u \in \mathbb{R}^m$, $y_c \in \mathcal{Q} \in \mathbb{R}^q$ is a continuous output, $y_d \in \mathbb{R}^q$ is a set of discrete measurements, and the independent variable $t \in \mathbb{R}$ denotes the time. The functions $f(x)$, $f_I(x) \in \mathbb{R}^n$ and $g(x) \in \mathbb{R}^{n\times m}$ are analytical vector fields, and the spaces $\mathcal{X}$ and $\mathcal{Q}$ are analytic manifolds.

Note that the first two equations of system (1) can be written alternatively as (Rivadeneira & Moog, 2012)

$$\dot{x}(t) = f(x(t)) + (f_1(x(t)) + g(x(t))u(t))\delta(t - \tau_k),$$
$$x(t_0) = x_0, \tag{2}$$

where $f_1 = f_I(x) - x$, and δ is the impulse applied at times τ_k, $k \in \mathbb{N}$. For the special case where $f_I(x) = x$, then (2) reduces to

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t)\delta(t - \tau_k), \tag{3}$$
$$x(t_0) = x_0. \tag{4}$$

Actually, the nonlinear ICS (1) is an autonomous system in the intervals $]\tau_{k-1}, \tau_k[$, $k = \{1, 2, \ldots\}$. For simplicity, assume that $t_0 = 0$, there is no impulse applied to the system in the interval $[0, \tau_1[$, and $u(\tau_i) = u_i$. Let $\Psi(t, 0, x_0)$ be a solution of the autonomous system[1] of the first equation in (1) for $t \in [0, \tau_1[$, *i.e.* $x(t) = \Psi(t, 0, x_0)$, $t \in [0, \tau_1[$. At $t = \tau_1$,

$$x(\tau_1^+) = f_I(x(\tau_1)) + g(x(\tau_1))u_1 \tag{5}$$
$$= f_I(\Psi(\tau_1, 0, x_0)) + g(\Psi(\tau_1, 0, x_0))\, u_1. \tag{6}$$

Now, for $t \in [\tau_1, \tau_2[$, where the first impulse has been already applied to the system, the state trajectory $x(t)$ is

$$\begin{aligned} x(t) &= \Psi(t, \tau_1, x(\tau_1^+)) \\ &= \Psi(t, \tau_1, f_I(\Psi(\tau_1, 0, x_0)) + g(\Psi(t, 0, x_0))\, u_1). \end{aligned}$$

In general, the state $x(t)$ in the interval $[\tau_{k-1}, \tau_k[$ follows the recursive equation

$$\begin{aligned} x(t) &= \Psi(t, \tau_{k-1}, x(\tau_{k-1}^+)), \quad t \in [\tau_{k-1}, \tau_k[, \\ x(\tau_k^+) &= f_I(x(\tau_k)) + g(x(\tau_k))u_k, \quad t = \tau_k, \ k \in \mathbb{N} \end{aligned} \tag{7}$$

where $\tau_0 = 0$, $x(\tau_0^+) = x_0$, and $k - 1$ impulses have been applied to the system. Note that $x(\cdot)$, and $g(\cdot)$ depend on x_0 and u_i.

If $f(x) = Ax$, $g(x) = B$, and $f_I(x) = A_I x$, this system is a linear ICS and can be expressed as (Medina & Lawrence, 2008)

$$\begin{cases} \dot{x}(t) = Ax(t), & x(0^+) = x_0, \quad t \neq \tau_k, \\ x(\tau_k^+) = A_I x(\tau_k) + Bu(\tau_k), & k \in \mathbb{N}, \\ y_c(t) = C_c x(t), & \text{or} \\ y_d(t) = C_d x(t), & \end{cases} \tag{8}$$

where A, B, A_I, and C_c (or C_d) have appropriate dimensions.

The state response for this class of systems can be generated explicitly as follows. Let us denote the final time as $t_f = \tau_{k+1}$, the set of time instants as $T = \{\tau_1, \ldots \tau_k\}$ such that Δ_i is equal to $\Delta_i = \tau_{i+1} - \tau_i$, and verifies that $\Delta_0 = \tau_1$, and $\Delta_k = t_f - \tau_k$. The state transition matrix of (8) is calculated recursively using (7) and results in $\Phi(t_f, 0) = e^{A\Delta_k} A_I e^{A\Delta_{k-1}} \cdots A_I e^{A\Delta_1} A_I e^{A\Delta_0}$.

The state transition matrix is invertible for all $t \in [0, t_f]$ if only if the matrix A_I is invertible, and in this case, $\Phi(0, t_f) = \Phi^{-1}(t_f, 0)$ (see Medina & Lawrence, 2008 for more details). The state response of system (8) on $[0, t]$ with k impulses applied to the system is $x(t) = \Phi(t, 0)x_0 + \sum_{j=1}^{k} \Phi(t, \tau_j)Bu_j$. Note that if $B = 0$ and $A_I = I$, the state transition matrix for LTI systems is recovered, that is, $\Phi(t, t_0) = e^{At}$ and the state response is just $x(t) = e^{At}x_0$. Now, if $B \neq 0$ but $A_I = I$, the state response equation becomes $x(t) = e^{At}\left(x_0 + \sum_{j=1}^{k} e^{-A\tau_j}Bu_j\right)$, which agrees with results in Yang (2001).

3. Observability for nonlinear impulsive systems

3.1. Strong observability

In standard nonlinear control systems (where the impulses are not involved), this property has been extensively developed, not only considering continuous outputs (Conte, Moog, & Perdon, 2007), but also discrete ones (Califano, Monaco, & Normand-Cyrot, 2003; Moral & Grizzle, 1995). A standard nonlinear control system with continuous output is called strongly observable, if the state can be deduced from the knowledge of the output and its time derivatives. For nonlinear ICS, the same notion will be maintained even if impulses are taken into account in the dynamics.

Definition 1. System (1) is said to be strongly observable at point $t = 0$, if there exist an integer n, and locally a function φ such that $x(0) = \varphi\left(y_c(0), \dot{y}_c(0), \ldots, y_c^{(n-1)}(0)\right)$.

Theorem 1. *System (1) is strongly observable at point $t = 0$, if and only if*

[1] The existence and uniqueness of the solution $\Psi(\cdot)$ is assumed. However, this is still an active field of research. See Ref. Lakshmikantham, Bainov, and Simeonov (1989) for an introduction.