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Robustness of Control Barrier Functions for Safety Critical Control ⋆

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Abstract: Barrier functions (also called certificates) have been an important tool for the verification of hybrid systems, and have also played important roles in optimization and multi-objective control. The extension of a barrier function to a controlled system results in a control barrier function. This can be thought of as being analogous to how Sontag extended Lyapunov functions to control Lypaunov functions in order to enable controller synthesis for stabilization tasks. A control barrier function enables controller synthesis for safety requirements specified by forward invariance of a set using a Lyapunov-like condition. This paper develops several important extensions to the notion of a control barrier function. The first involves robustness under perturbations to the vector field defining the system. Input-to-State stability conditions are given that provide for forward invariance, when disturbances are present, of a "relaxation" of set rendered invariant without disturbances. A control barrier function can be combined with a control Lyapunov function in a quadratic program to achieve a control objective subject to safety guarantees. The second result of the paper gives conditions for the control law obtained by solving the quadratic program to be Lipschitz continuous and therefore to gives rise to well-defined solutions of the resulting closed-loop system.

Keywords: Barrier function, Invariant set, Quadratic program, Robustness, Continuity

1. INTRODUCTION

Lyapunov functions are used to certify stability properties of a set without calculating the exact solution of a system. In a similar manner, barrier certificates (functions) are used to verify temporal properties (such as safety, avoidance, eventuality) of a set, without the difficult task of computing the system's reachable set; see Prajna and Rantzer (2007), Prajna et al. (2007). These same references show that when the vector fields of the system are polynomial and the sets are semi-algebraic, barrier certificates can be computed by sum-of-squares optimization. In the original formulation of Prajna et al. (2007), all sublevel sets of the barrier certificate were required to be invariant because the derivative of the barrier certificate along solutions was required to be non-positive. This condition was relaxed by Kong et al. (2013) and Dai et al. (2013) so that tighter over-approximations of the reachable set could be obtained, and such that more expressive barrier certificates could be synthesized using semi-definite programming. The key idea there was to only require that a single sublevel set be invariant, namely, the set of points where the barrier certificate was non-positive.

The natural extension of barrier functions to a system with control inputs is a control barrier function (CBF), first proposed by Wieland and Allgöwer (2007); this work used the original condition of a barrier function that imposes invariance of all sublevel sets. The unification of control Lyapunov functions (CLFs) with CBFs appeared at the same conference in Romdlony and Jayawardhana (2014) and Ames et al. (2014b), using two contrasting formulations. The objective of Romdlony and Jayawardhana (2014) was to incorporate into a single feedback law the conditions required to simultaneously achieve asymptotic stability of an equilibrium point, while avoiding an unsafe set. The feedback law was constructed using Sontag's universal control formula (Sontag (1989)), provided that a "control Lyapunov barrier function" inequality could be met. Importantly, if the stabilization and safety objectives were in conflict, then no feedback law could be proposed. In contrast, the approach of Ames et al. (2014b) was to pose a feedback design problem that *mediates* the safety and stabilization requirements, in the sense that safety is

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always guaranteed, and progress toward the stabilization objective is assured when the two requirements "are not in conflict".

The essential difference between these two approaches is perhaps best understood through an example. A vehicle equipped with Adaptive Cruise Control (ACC) seeks to converge to and maintain a fixed cruising speed, as with a common cruise control system. Converging to and maintaining fixed speed is naturally expressed as asymptotic stabilization of a set. With ACC, the vehicle must in addition guarantee a safety condition, namely, when a slower moving vehicle is encountered, the controller must automatically reduce vehicle speed to maintain a guaranteed lower bound on time headway or following distance, where the distance to the leading vehicle is determined with an onboard radar. When the leading car speeds up or leaves the lane, and there is *no longer a conflict between safety and desired cruising speed*, the adaptive cruise controller automatically increases vehicle speed. The time-headway safety condition is naturally expressible as a control barrier function. In the approach of Ames et al. (2014b), a Quadratic Program (QP) mediates the two inequalities associated with the CLFs and CBFs; in particular, relaxation is used to make the stability objective a soft constraint while safety is maintained as a hard constraint. In this way, safety and stability do not need to be simultaneously satisfiable. On the other hand, the approach of Romdlony and Jayawardhana (2014) is only applicable when the two objectives can be simultaneously met.

A second, although less important, difference in the two approaches is that Romdlony and Jayawardhana (2014) used the more restrictive invariance condition of Prajna and Rantzer (2007), while Ames et al. (2014b) used the relaxed condition of Kong et al. (2013), appropriately interpreted for the type of barrier function often used in optimization, see Boyd and Vandenberghe (2004), where the barrier function is unbounded on the boundary of the allowed set, instead of vanishing on the set boundary.

The present paper builds on previous work in two important directions. First, the robustness of barrier functions and control barrier functions under model perturbation is investigated. An Input-to-State (ISS) stability property of a safe set is established when perturbations are present and the barrier function vanishes on the set boundary. The second result gives conditions that guarantee local Lipschitz continuity of the feedback law arising from the QP used to mediate safety and asymptotic convergence to a set. The analysis is based on the constraint qualification conditions along with the KKT conditions for optimality. While the result is applicable to the type of barrier function in Ames et al. (2014b), it will be stated for barrier functions used in this paper that vanish on the set boundary.

The remainder of the paper is organized as follows. Section 2 defines zeroing barrier functions and zeroing control barrier functions, and establishes a robustness property to model perturbations. Section 3 develops the conditions for the solution of the QP to be locally Lipschitz continuous in the problem data. The theory developed is illustrated in Section 4 on adaptive cruise control. Section 5 summarizes the conclusions.

Notation: The set of real, positive real and non-negative real numbers are denoted by $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$, respectively. The Euclidean norm is denoted by $\|\cdot\|$. The transpose of matrix A is denoted by $A^\top$. The interior and boundary of a set $\mathcal{S}$ are denoted by $\mathrm{Int}(\mathcal{S})$ and $\partial\mathcal{S}$, respectively. The distance from x to a set $\mathcal{S}$ is denoted by $\|x\|_{\mathcal{S}} = \inf_{s\in\mathcal{S}} \|x-s\|$. For any essentially bounded function $g : \mathbb{R} \to \mathbb{R}^n$, the infinity norm of g is denoted by $\|g\|_\infty = \mathrm{ess}\sup_{t\in\mathbb{R}} \|g(t)\|$.

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called *Lipschitz continuous* on $I \subset \mathbb{R}^n$ if there exists a constant $L \in \mathbb{R}^+$ such that $\|f(x_2) - f(x_1)\| \le L\|x_2 - x_1\|$ for all $x_1, x_2 \in I$, and called *locally Lipschitz continuous* at a point $x \in \mathbb{R}^n$ if there exist constants $\delta \in \mathbb{R}^+$ and $M \in \mathbb{R}^+$ such that $\|f(x) - f(x')\| \le M\|x - x'\|$ holds for all $\|x - x'\| \le \delta$. A continuous function $\beta_1 : [0, a) \to [0, \infty)$ for some $a > 0$ is said to belong to *class* $\mathcal{K}$ if it is strictly increasing and $\beta_1(0) = 0$. A continuous function $\beta_2 : [0, b) \times [0, \infty) \to [0, \infty)$ for some $b > 0$ is said to belong to *class* $\mathcal{KL}$, if for each fixed s, the mapping $\beta_2(r, s)$ belongs to class $\mathcal{K}$ with respect to r and for each fixed r, the mapping $\beta_2(r, s)$ is decreasing with respect to s and $\beta_2(r, s) \to 0$ as $s \to \infty$.

2. ZEROING (CONTROL) BARRIER FUNCTIONS

The barrier function and control barrier function considered in this paper are based on Kong et al. (2013), Dai et al. (2013), and Wieland and Allgöwer (2007). As in Ames et al. (2014b), the primary focus is to establish forward invariance of a given set $\mathcal{C}$, which one may interpret as an under approximation of the "initial set" and the "safe set" in previous formulations of barrier functions. The main contribution of the section is a robustness property under model perturbations.

Consider a nonlinear system on $\mathbb{R}^n$,

$$\dot{x} = f(x), \tag{1}$$

with f locally Lipschitz continuous. Denote by $x(t, x_0)$ the solution of (1) with initial condition $x_0 \in \mathbb{R}^n$. To simplify notation, the solution is also denoted by $x(t)$ whenever the initial condition does not play an important role in the discussion. The *maximal interval of existence* of $x(t, x_0)$ is denoted by $I(x_0)$. When $I(x_0) = \mathbb{R}_0^+$ for any $x_0 \in \mathbb{R}^n$, the differential equation (1) is said to be *forward complete*. A set $\mathcal{S}$ is called *forward invariant* if for every $x_0 \in \mathcal{S}$, $x(t, x_0) \in \mathcal{S}$ for all $t \in I(x_0)$.

For $\epsilon \ge 0$, define the family of closed sets $\mathcal{C}_\epsilon$ as

$$\mathcal{C}_\epsilon = \{x \in \mathbb{R}^n : h(x) \ge -\epsilon\}, \tag{2}$$

$$\partial\mathcal{C}_\epsilon = \{x \in \mathbb{R}^n : h(x) = -\epsilon\}, \tag{3}$$

$$\mathrm{Int}(\mathcal{C}_\epsilon) = \{x \in \mathbb{R}^n : h(x) > -\epsilon\}, \tag{4}$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function. By construction, $\mathcal{C}_{\epsilon_1} \subset \mathcal{C}_{\epsilon_2}$ for any $\epsilon_2 > \epsilon_1 \ge 0$. For simplicity, the set $\mathcal{C}_0$ is denoted by $\mathcal{C}$.

The definition of a barrier function is made easier through an appropriate extension of the notion of class $\mathcal{K}$ function.

Definition 1. (Based on Khalil (2002)) A continuous function $\beta : (-b, a) \to (-\infty, \infty)$ for some $a, b > 0$ is said to