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Uncertain saddle point equilibrium differential games with non-anticipating strategies

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ABSTRACT

In this paper, we investigate uncertain saddle point equilibrium differential games under uncertain environment. We propose an optimistic value game model and define the value function of the game by introducing the concept of non-anticipating strategy. We prove the continuity and dynamic programming property of the value function. Then we derive the uncertain Hamilton–Jacobi–Isaacs equation by the viscosity solution approach.

1. Introduction

In the past few decades, game theory has been an active research field in operations research and control theory. Von Meumann and Morgenstern [24] first established the modern game theory. Later, Isaacs [15] studied a two person zero sum differential game model in a dynamical system which initiated the research of differential game theory. Pontryagin [25] considered a class of differential games with the maximum principle theorem. Friedman [12] and Berkovitz [2] investigated the existence theorem and approximation method for differential games.

In previous work, one important way to solve the two person zero sum differential game is to transform the original problem to solving a PDE (called HJI equaiton). An important assumption is that the value function of the game is assumed to be sufficiently smooth (e.g. twice differentiable) to make sense of the related HJI equation. Nevertheless, this assumption is usually impossible to be achieved. Many researchers [5,9,11] has worked on this difficulty with some relaxed assumptions. The breakthrough is the establishment of concept of non-anticipating strategy and viscosity solution (see e.g. [6,8,10,18]).

Stochastic differential game [1,3,7,14,28] also received much attention. However, noises in some particular dynamical systems do not behave like randomness, such as the price of new stock, bridge strength and oil field reserves. There are no enough samples to ensure the estimated probability distribution of the noises is close enough to the long-run cumulative frequency. Hence, stochastic differential equations are not able to appropriately model these dynamical systems [23]. To estimate this kind of indeterministic noises, people have to invite some domain experts to evaluate the belief degree that each event may occur. Liu [19] founded uncertainty theory in 2007 to rationally deal with personal belief degrees. Nowadays, uncertain theory has been applied to many fields (see e.g. [4,26,27,30,34,35]). Thus, for those dynamical systems which cannot be appropriately described by stochastic differential equations, we may use uncertain differential equations. Two person zero sum uncertain differential games were analyzed (see e.g. [13,29,31,32]).

In this paper, we consider a two person zero sum uncertain differential game with non-anticipating strategies. The rest of this paper is organized as follows. In Section 2, we review some basic concepts about uncertainty theory. In Section 3, we formulate our saddle point equilibrium game model and introduce the non-anticipating strategy. In Section 4, we discuss the properties of the value function. In the Section 5, we establish the relationship between the value function and the uncertain HJI equation with the viscosity solution. In the last section, we give an example to illustrate our results.

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2. Preliminaries

In this section, we introduce some important basic concepts about uncertainty theory, which are used throughout the paper.

Uncertainty theory is a branch of axiomatic mathematics to deal with human uncertainty arising from the belief degrees. Let Γ be a nonempty set and $\mathcal{L}$ be a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. Then uncertain measure $\mathcal{M}\{\Lambda\}$ is used to evaluate the belief degree that each event Λ may occur. The axiomatic definition is as follows.

Definition 1. [19] A set function $\mathcal{M}$ defined on the σ-algebra $\mathcal{L}$ is called an uncertain measure if it satisfies three axioms:

Axiom 1. (Normality) $\mathcal{M}\{\Gamma\} = 1$;

Axiom 2. (Duality) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$, for $\Lambda \in \mathcal{L}$;

Axiom 3. (Countable Subadditivity) $\mathcal{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$, for $\Lambda_i \in \mathcal{L}, i = 1, 2, \ldots$.

Then the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. The product uncertain measure $\mathcal{M}$ is an uncertain measure on the σ-algebra $\mathcal{L}_1 \times \mathcal{L}_2 \times \cdots$ satisfying $\mathcal{M}\{\prod_{k=1}^{\infty} \Lambda_k\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$. Based on the above axioms, the concepts of uncertain variable uncertainty distribution, independence, uncertain process, etc. are defined in Liu [22].

Definition 2. [19] An uncertain variable ξ is called a normal uncertain variable if its distribution function is

$$\Phi(x) = \left[1 + \exp\left(\frac{\pi(e - x)}{\sqrt{3}\sigma}\right)\right]^{-1}, x \in \mathbb{R},$$

where e is the expected value and σ^2 is the variance with $\sigma > 0$. We can denote ξ by $\mathcal{N}(e, \sigma)$.

Definition 3. [19] Assume that ξ is an uncertain variable, and $\alpha \in (0, 1]$. Then $\xi_{\sup}(\alpha) = \sup\{r | \mathcal{M}\{\xi \ge r\} \ge \alpha\}$ is called the α-optimistic value to ξ.

Theorem 1. *[22] Let ξ be an uncertain variable with uncertainty distribution function Φ. Then its α-optimistic value can be calculated by*

$$\xi_{\sup}(\alpha) = \Phi^{-1}(1 - \alpha).$$

Definition 4. [21] An uncertain process C_t is called a Liu process if it satisfies:

(1) $C_0 = 0$ and almost all sample paths are Lipschitz continuous.
(2) C_t has stationary and independent increments.
(3) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2.

Definition 5. [20] Let C_t be a Liu process, and f_1, f_2 be some given functions. Then

$$\mathrm{d}X_t = f_1(t, X_t)\mathrm{d}t + f_2(t, X_t)\mathrm{d}C_t \tag{1}$$

is called an uncertain differential equation.

To get a full understanding of the concepts in uncertainty theory such as uncertain calculus, uncertain vector, multidimensional uncertain differential equation, please refer to Liu [23].

Theorem 2. *[33] Let C_t be a Liu process. Then there exists an uncertain variable such that $K(\gamma)$ is the Lipschitz constant of the sample path $C_t(\gamma)$ for each γ, and*

$$\mathcal{M}\{K \le x\} \ge 2\Phi(x) - 1,$$

where $\Phi(x)$ is the distribution function of standard normal uncertain variable $\mathcal{N}(0, 1)$.

3. Problem formulation

In this section, we formulate our uncertain saddle point equilibrium differential game model and provide some estimates about the system states. In addition, we introduce the notion of non-anticipating strategy.

We consider a control system described by an uncertain differential equation as follows:

$$\begin{cases} \mathrm{d}\boldsymbol{X}_s = \boldsymbol{f}(s, \boldsymbol{X}_s, \boldsymbol{u}_1, \boldsymbol{u}_2)\mathrm{d}s + \boldsymbol{g}(s, \boldsymbol{X}_s, \boldsymbol{u}_1, \boldsymbol{u}_2)\mathrm{d}\boldsymbol{C}_s, & t \le s \le T, \\ \boldsymbol{X}_t = \boldsymbol{x}, \end{cases} \tag{2}$$

where $\boldsymbol{f} : [0, T] \times \mathbb{R}^n \times \mathbb{U}_1 \times \mathbb{U}_2 \to \mathbb{R}^n$, $\boldsymbol{g} : [0, T] \times \mathbb{R}^n \times \mathbb{U}_1 \times \mathbb{U}_2 \to \mathbb{R}^{n\times k}$, with $\mathbb{U}_1 \in \mathbb{R}^p, \mathbb{U}_2 \in \mathbb{R}^p$ being some non-empty closed convex sets. In the above system, $\boldsymbol{X}_s$ is the state vector, t is the initial time, $\boldsymbol{x}$ is the initial state, $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are control vectors taken by two involved persons. We label them as Player 1 and Player 2 for convenience. For Player $i (i = 1, 2)$, we denote the admissible control set by $\mathcal{U}_i[t, T] = \{\boldsymbol{u}_i : [t, T] \to \mathbb{U}_i | \boldsymbol{u}_i$ is measurable$\}$. In addition, $\boldsymbol{C}_s$ is a k-dimensional uncertain Liu process. Then we have the following assumption:

(**S**) The map $\boldsymbol{f}$ and $\boldsymbol{g}$ are continuous and for any $(s, u_1, u_2) \in [0, T] \times \mathbb{U}_1 \times \mathbb{U}_2$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, there exists a constant L such that

$$\begin{cases} |\boldsymbol{f}(s, \boldsymbol{x}, u_1, u_2) - \boldsymbol{f}(s, \boldsymbol{y}, u_1, u_2)| + |\boldsymbol{g}(s, \boldsymbol{x}, u_1, u_2) \\ \quad -\boldsymbol{g}(s, \boldsymbol{y}, u_1, u_2)| \le L|\boldsymbol{x} - \boldsymbol{y}|, \\ |\boldsymbol{f}(s, \boldsymbol{x}, u_1, u_2)| + |\boldsymbol{g}(s, \boldsymbol{x}, u_1, u_2)| \le L(1 + |\boldsymbol{x}|), \end{cases}$$

where $|\boldsymbol{x}| = \max_{1\le i\le n} |x_i|$ for vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $|\boldsymbol{A}| = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|$ for matrix $\boldsymbol{A} = (a_{ij})_{m\times n}$.

Lemma 1. *Under assumption (**S**), for any initial pair $(t, \boldsymbol{x}) \in [0, T] \times \mathbb{R}^n$ and any admissible control $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ in $\mathcal{U}_1[t, T] \times \mathcal{U}_2[t, T]$, there exists a unique solution $\boldsymbol{X}_t$ to the uncertain system (2). Moreover, for any event $\gamma \in \Gamma$, we have the following estimates:*

$$\begin{cases} |\boldsymbol{X}_s(\gamma)| \le e^{L(1+K(\gamma))(s-t)}(|\boldsymbol{x}| + 1) - 1, & (3) \\ |\boldsymbol{X}_s(\gamma) - \boldsymbol{x}| \le (e^{L(1+K(\gamma))(s-t)} - 1)(|\boldsymbol{x}| + 1), & (4) \end{cases}$$

and

$$|\boldsymbol{X}_s(\gamma) - \hat{\boldsymbol{X}}_s(\gamma)| \le |\boldsymbol{x} - \hat{\boldsymbol{x}}| e^{L(1+K(\gamma))(s-t)}, \tag{5}$$

where $\boldsymbol{X}_s$ and $\hat{\boldsymbol{X}}_s$ are two solutions to the uncertain differential equation with different initial values.

Proof. When assumption (**S**) holds, according to Theorem 4 in Ji and Zhou [17], there exists a unique solution to uncertain differential Eq. (2). Next, we only prove the first estimate since the others can be obtained similarly. For each event γ, we have

$$\boldsymbol{X}_s(\gamma) = \boldsymbol{x} + \int_t^s \boldsymbol{f}(r, \boldsymbol{X}_r(\gamma), \boldsymbol{u}_1, \boldsymbol{u}_2)\mathrm{d}r + \int_t^s \boldsymbol{g}(r, \boldsymbol{X}_r(\gamma), \boldsymbol{u}_1, \boldsymbol{u}_2)\mathrm{d}\boldsymbol{C}_r(\gamma).$$

Then, assumption (S) and properties of uncertain calculus yield

$$\begin{aligned} |\boldsymbol{X}_s(\gamma)| &\le |\boldsymbol{x}| + L\int_t^s (1 + |\boldsymbol{X}_r(\gamma)|)\mathrm{d}r + K(\gamma)L\int_t^s (1 + |\boldsymbol{X}_r(\gamma)|)\mathrm{d}r \\ &= |\boldsymbol{x}| + L(1 + K(\gamma))(s - t) + L(1 + K(\gamma))\int_t^s |\boldsymbol{X}_r(\gamma)|\mathrm{d}r. \end{aligned}$$

By Gronwall's inequality, we have

$$\begin{aligned} |\boldsymbol{X}_s(\gamma)| &\le |\boldsymbol{x}| e^{\int_t^s L(1+K(\gamma))\mathrm{d}\tau} + \int_t^s L(1 + K(\gamma)) e^{\int_r^s L(1+K(\gamma))\mathrm{d}\tau}\mathrm{d}r \\ &= e^{L(1+K(\gamma))(s-t)}(|\boldsymbol{x}| + 1) - 1. \end{aligned}$$

Remark 1. Note that the estimates hold for each $\gamma \in \Gamma$ which is different from both deterministic case or stochastic case.

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