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IFAC PapersOnLine 50-1 (2017) 431–437

Three examples of the stability properties of the invariant extended Kalman filter ⋆

A. Barrau * **S. Bonnabel** **

* *SAFRAN TECH, Groupe Safran, Rue des Jeunes Bois - Châteaufort, 78772 Magny Les Hameaux CEDEX.*
** *MINES ParisTech, PSL Reasearch University, Centre for Robotics, 60 bd Saint-Michel, 75006 Paris, France.*

Abstract: In the aerospace industry the (multiplicative) extended Kalman filter (EKF) is the most common method for sensor fusion for guidance and navigation. However, from a theoretical point of view, the EKF has been shown to possess local convergence properties only under restrictive assumptions. In a recent paper, we proved a slight variant of the EKF, namely the invariant extended Kalman filter (IEKF), *when used as a nonlinear observer*, possesses local convergence properties under the same assumptions as those of the linear case, for a class of systems defined on Lie groups. This is especially interesting as the IEKF also retains all the desirable features of the standard EKF, especially its relevant tuning in the presence of noises. In the present paper we provide three examples of engineering interest where the theory is shown to apply, yielding three EKF-like algorithms with guaranteed local convergence properties. Beyond those contributions, the present article is sufficiently accessible to help the practitioner understand through concrete examples the general IEKF theory, and to provide him with guidelines for the design of IEKFs.

Keywords: Estimation, Kalman filtering, nonlinear systems, Lie groups, navigation.

1. INTRODUCTION

In the aerospace industry the (multiplicative) extended Kalman filter (EKF) is the most popular method for sensor fusion for guidance and navigation. However, from a theoretical point of view, the EKF has been shown to possess local convergence properties only under restrictive assumptions Song and Grizzle (1995); Krener (2003), and as a matter of fact it can actually diverge, even for small initial errors. The recent paper Barrau and Bonnabel (2017) proves a slight variant of the EKF, namely the invariant extended Kalman filter (IEKF), possesses local convergence properties under highly reasonable assumptions for a well characterized class of systems defined on Lie groups. The IEKF was originally introduced in Bonnabel (2007); Bonnabel et al. (2009b), and builds upon the theory of symmetry-preserving observers Bonnabel et al. (2009a). It can also be related to the generalized multiplicative EKF of Martin and Salaün (2010), the discrete EKF on Lie groups Bourmaud et al. (2013), see also de Ruiter and Forbes (2016).

The principles of the IEKF theory as presented in Barrau and Bonnabel (2017) are not easy to grasp. The main goal of the present paper is thus to provide a user friendly presentation and discussion of the IEKF as described in Barrau and Bonnabel (2017), and to illustrate its stability properties on three examples of engineering interest. Even though the purpose of the present article is essentially tutorial, it also contains novel theoretical results as we derive three non-linear filters for three examples of engineering interest, and guarantee stability of the derived filters.

Those examples could certainly be tackled through non-linear observers, along the lines of e.g., Hua et al. (2014); Wolfe et al. (2011); Batista et al. (2014); Izadi and Sanyal (2014); Sanyal and Nordkvist (2012); Zamani et al. (2014); Lee et al. (2007); Hua et al. (2016); Tayebi et al. (2007), and (almost) global convergence properties could be - or have already been - obtained. The interest (and difference) of our approach with respect to the non-linear observer literature though, is that the three non-linear proposed filters (namely IEKFs) 1- accomodate discrete time measurements with arbitrary and varying sampling times, 2- the gain tuning matches the modeled variance of the noises through (linearized) Kalman's theory 3- this implies the gains easily accomodate time-varying features, such as time-varying covariance matrices, 4- contrarily to non-linear observers, the filter provides an indication (through the covariance matrix P_t) of the extent of uncertainty conveyed by the estimate and 5- the filters viewed as observers, that is, when noise is turned off, converge around *any* trajectory, with an attraction radius which is uniform over time. It is worthy to note that, to that respect, the three filters achieve the same goals as the ones pursued by the very recent XKF Johansen and Fossen (2016), albeit a wholly different method. Note though, it has not yet been shown an XKF may be built on the following examples.

In a nutshell, the IEKFs proposed here should be appealing to the aerospace engineers: they retain all the characteristics of the standard EKF (first-order optimality, realtive ease of tuning, adaptivity to time-varying features

⋆ This work is supported by the company Safran.

10.1016/j.ifacol.2017.08.061

and to discrete aperiodic measurements), but with additional stability properties, when studied in a deterministic setting using the tools of dynamical systems theory.

The paper is organized as follows. Section II is a concise tutorial summary and discussion on Barrau and Bonnabel (2017). Each following section deals with an example. A conclusion seemed not necessary, so it was omitted.

2. REVIEW OF THE IEKF METHODOLOGY AND CONVERGENCE PROPERTIES

In this section, we review the IEKF methodology as presented in Barrau and Bonnabel (2017), that is, for continuous time dynamics with discrete time observations for systems defined on Lie groups. The exposure is meant to be concise and tutorial, and is enhanced by discussions. See the Appendix for more details on matrix Lie groups.

2.1 Considered class of systems and IEKF equations

Consider in this section a dynamics on a matrix Lie group $G \subset \mathbb{R}^{N\times N}$ with state $\chi_t \in G$ satisfying:

$$\frac{d}{dt}\chi_t = f_{u_t}(\chi_t) + \chi_t w_t, \tag{1}$$

where w_t is a continuous white noise belonging to the Lie algebra $\mathfrak{g}$ (see the Appendix). Let $q = \dim G$ denote the dimension of the Lie group G (or alternatively defined by $q = \dim\, \mathfrak{g}$). Assume moreover the following relation holds

$$f_u(ab) = af_u(b) + f_u(a)b - af_u(Id)b \tag{2}$$

for all $(u, a, b) \in U \times G \times G$. This system can be associated with two different kinds of discrete observations at arbitrary times $t_0 < t_1 < t_2, \cdots$.

Left-invariant observations The first family of outputs we are interested in write:

$$Y^1_{t_n} = \chi_{t_n}\left(d^1 + B^1_n\right) + V^1_n, ..., Y^k_{t_n} = \chi_{t_n}\left(d^k + B^k_n\right) + V^k_n, \tag{3}$$

where $(d^i)_{i\leq k}$ are known vectors of $\mathbb{R}^N$, and where the $(V^i_n)_{i\leq k}$, $(B^i_n)_{i\leq k}$ are centered Gaussian varibables noises with known covariance matrices.

The outputs are said to be "left-invariant" as, in the absence of noise, the outputs are of the form $h(\chi) = \chi d$ so that, $\chi_2 h(\chi_1) = h(\chi_2\chi_1)$. This property is also referred to as left equivariance in the mathematics literature and in the theory of symmetry-preserving observers. For left-invariant observations, a Left-Invariant EKF (LIEKF) should always be used.

The Left-Invariant Extended Kalman Filter (LIEKF) is defined through the usual following propagation and update steps:

$$\frac{d}{dt}\hat{\chi}_t = f_{u_t}(\hat{\chi}_t), \quad t_{n-1} \leq t < t_n, \qquad \text{Propagation} \tag{4}$$

$$\hat{\chi}^+_{t_n} = \hat{\chi}_{t_n} \exp\left[L_n \begin{pmatrix} \hat{\chi}^{-1}_{t_n} Y^1_{t_n} - d^1 \\ \cdots \\ \hat{\chi}^{-1}_{t_n} Y^k_{t_n} - d^k \end{pmatrix}\right], \quad \text{Update} \tag{5}$$

where the function $L_n : \mathbb{R}^{kN} \to \mathbb{R}^q$ is defined through linearizations as in the conventional EKF theory. But here, instead of considering the usual linear state error $\hat{\chi}_t - \chi_t$, one must consider the following left-invariant error between true state χ_t and the estimated state $\hat{\chi}_t$:

$$\eta^L_t = \chi_t^{-1}\hat{\chi}_t. \tag{6}$$

which is the counterpart of the linear error $\hat{\chi}_t - \chi_t$ (which has no proper meaning in the present context), when dealing with a state space that is a Lie group. Note that, this error is nominally equal to identity matrix and not zero. The rationale of the IEKF theory, and more generally the theory of symmetry-preserving observers, is to linearize the error system at the propagation and update state. It turns out the error system, with an error defined this way, has remarkable properties, that are key to prove the IEKF stability properties of Barrau and Bonnabel (2017).

Right-invariant observations The second family of observations we are interested in have the form:

$$Y^1_{t_n} = \chi^{-1}_{t_n}\left(d^1 + V^1_n\right) + B^1_n, ..., Y^k_{t_n} = \chi^{-1}_{t_n}\left(d^k + V^k_n\right) + B^k_n. \tag{7}$$

with the same notation as in the previous paragraph. The Right-Invariant EKF (RIEKF), always to be used for right-invariant observations of the form (7) is defined here in the same way, alternating between continous time propagation and discrete time update steps:

$$\frac{d}{dt}\hat{\chi}_t = f_{u_t}(\hat{\chi}_t), \quad t_{n-1} \leq t < t_n, \tag{8}$$

$$\hat{\chi}^+_{t_n} = \exp\left[L_n \begin{pmatrix} \hat{\chi}_{t_n} Y^1_{t_n} - d^1 \\ \cdots \\ \hat{\chi}_{t_n} Y^k_{t_n} - d^k \end{pmatrix}\right] \hat{\chi}_{t_n}. \tag{9}$$

To tune the gain L_n the state error must be linearized, but in this case we rather consider the right-invariant error

$$\eta^R_t = \hat{\chi}_t \chi_t^{-1}. \tag{10}$$

Gain tuning To tune the gain matrix L_n, one must linearize the error equation associated to (6), or respectively (10). To do so, the user can refer to the general theory of Barrau and Bonnabel (2017), or rather proceed to a case by case derivation as done in the examples below, which is recommended. In any case, one can associate to the nonlinear error (6), or (10), a vector $\xi_t \in \mathbb{R}^q$ that captures the error up to the first order. It can be used to obtain a linear approximation to the true error system, of the form:

$$\frac{d}{dt}\xi_t = A_t\xi_t + D(\hat{\chi}_t)\tilde{w}_t \tag{11}$$

where $\tilde{w}_t$ is a continuous noise in $\mathbb{R}^q$ and to a linearized error update equation of the form

$$\xi^+_{t_n} = \xi_{t_n} - L_n(H\xi_{t_n} + E(\hat{\chi}_{t_n})V_n) \tag{12}$$

with V_n a centered Gaussian. To account for the fact the stochastic terms entering the system depend on the estimated trajectory, we define as in the standard EKF theory with non-additive noises (see e.g. Stengel (1986)) the covariance matrices

$$Q(\hat{\chi}_t) = D(\hat{\chi}_t)Cov(\tilde{w}_t)D(\hat{\chi}_t)^T$$
$$N(\hat{\chi}_{t_n}) = E(\hat{\chi}_{t_n})Cov(V_n)E(\hat{\chi}_{t_n})^T$$

As in the standard EKF methodology, the "optimal" gain L_n is then obtained through the Kalman equations: