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Directed cycles and multi-stability of coherent dynamics in systems of coupled nonlinear oscillators

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Abstract: We analyse the dynamics of networks of coupled nonlinear systems in terms of both topology of interconnections as well as the dynamics of individual nodes. Here we focus on two basic and extremal components of any network: chains and cycles. In particular, we investigate the effect of adding a directed feedback from the last element in a directed chain to the first. Our analysis shows that, depending on the network size and internal dynamics of isolated nodes, multiple coherent and orderly dynamic regimes co-exist in the state space of the system. In addition to the fully synchronous state an attracting rotating wave solution occurs. The basin of attraction of this solution apparently grows with the number of nodes in the loop. The effect is observed in networks exceeding a certain critical size. Emergence of the attracting rotating wave solution can be viewed as a "topological bifurcation" of network dynamics in which removal or addition of a single connection results in dramatic change of the overall coherent dynamics of the system.

Keywords: Coupled systems, reaction network, eigenvalue, synchronization, wave solutions

1. INTRODUCTION

Networks of interconnected and interacting systems have been in the focus of attention for many decades. Substantial progress has been made towards understanding of some of the facets of their collective behavior such as e.g. synchronization (see e.g. (Pikovsky et al., 2001), (Strogatz, 2003) and references therein). Various authors reported that under certain conditions, a large class of nonlinear dynamical systems may exhibit globally asymptotically stable synchronization (Pogromsky, 1998), (Scardovi et al., 2010) as well as the partial one (Pogromsky et al., 2002), Belykh et al. (2000), (Belykh et al., 2004). Emergence and asymptotic properties of these collective behaviors has been shown to depend crucially on the particular network topologies (Chandrasekar et al., 2014). Yet, the link between network topology, dynamics of individual network nodes, and the overall collective network dynamics is not completely understood.

Here we look at two basic and extremal topological ingredients of complex networks: a chain and a cycle. Our motivation to focus on these two very basic configurations is that despite simplicity of these configurations, many complex physical networks can be viewed as an interconnection of the two (Gorban et al., 2010). Furthermore, recent computational studies revealed that cycles on their own could be important for explaining sustained coherent oscillatory network activity (Garcia et al., 2014).

We begin our investigation by analysing the dynamics of a system of coupled neutrally stable linear equations. The dynamics is essentially governed by a coupling matrix that corresponds to directed interconnections in the system. The equations can also be viewed as a model describing dynamics of damped oscillations in kinetic systems, and thus in what follows we refer to it as such. The results are provided in Section 2. In Section 3 we consider a more generalized setting, in which the dynamics of each individual node is governed by a nonlinear, albeit semi-passive (Pogromsky, 1998), oscillator. The equations describing oscillators in each node are of the FizHugh-Nagumo (FHN) type (FitzHugh, 1961). These oscillators are, in turn, an adaptation of the van der Pol oscillator. We found that

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for sufficiently long directed chains, adding feedback from the last node to the first one gives rise to the existence of rotating waves. Section 4 contains a discussion of our findings, and Section 5 concludes the paper.

2. COUPLED NEUTRALLY STABLE SYSTEMS

Consider the following system of linear first-order differential equations:

$$\dot{P} = KP, \tag{1}$$

where $P = \mathrm{col}(p_1, p_2, \dots, p_n) \in \mathbb{R}^n$, and matrix $K = (k_{ij})$ is defined as follows:

$$k_{ij} = \begin{cases} q_{ij} & \text{if } i \neq j; \\ -\sum\limits_{m,\, m\neq i} q_{mi} & \text{if } i = j. \end{cases} \tag{2}$$

Note that K is a Metzler matrix [1] with zero column sums. Off-diagonal elements k_{ij}, $i \neq j$, of the matrix K can be viewed as the connection weights between the i-th and the j-th nodes in the network. The matrix K can be related to the Laplacian matrix L (see e.g. (Bollobas, 1998)) of an associated directed network in which the overall connectivity pattern is the same except for that the direction of all connections is altered. The Laplacian for the latter network is thus $L = -K^T$. Note, however, that this relation does not necessarily hold for the original network.

Consider the simple simplex

$$\Delta_n = \left\{ P | p_i \geq 0, \sum\nolimits_i p_i = 1 \right\}.$$

Δ_n is clearly forward invariant under the dynamics (1) since it preserves non-negativity and obeys the "conservation law" $\sum_i p_i = \mathrm{const}$. (The latter follows immediately from the fact that K has zero column sums.) Thus any solution $P(\cdot; t_0, P_0)$ of (1) starting from $P_0 = P(t_0) \in \Delta_n$ remains in Δ_n for all $t \geq t_0$.

The invariance of Δ_n under (1) can be used to prove certain important properties of K and its associated system (1). Two examples are presented below.

- **Equilibria.** The Perron-Frobenius theorem implies the existence of a non-negative vector P^* such that $KP^* = 0$, i.e. P^* defines an equilibrium of system (1). The existence of this vector P^* can also be deduced from the forward invariant of Δ_n. Indeed, as any continuous map $\Phi : \Delta_n \to \Delta_n$ has a fixed point (Brouwer fixed point theorem), $\Phi = \exp(Kt)$ has a fixed point in Δ_n for any $t \geq t_0$. If $\exp(Kt)P^* = P^*$ for some $P^* \in \Delta_n$ and sufficiently small $t > t_0$, then $KP^* = 0$ because
$$\exp(Kt)P = P + tKP + o(t^2).$$
- **Eigenvalues of K.** It is clear that K has a zero eigenvalue. In fact, Gerschgorin's theorem states that all eigenvalues of K are in the union of closed discs
$$D_i = \{\lambda \in \mathbb{C} | \|\lambda - k_{ii}\| \leq |k_{ii}|\}.$$
Thus K does not have purely imaginary eigenvalues. This fact can also be deduced from the forward invariance of Δ_n in combination with the assumption of a positive equilibrium P^*. We exclude the eigenvector corresponding to the zero eigenvalue and consider K on the invariant hyperplane where $\sum_i p_i = 0$. If K has a purely imaginary eigenvalue λ, then there exists a $2D$ K-invariant subspace U, where K has two conjugated imaginary eigenvalues, λ and $\overline{\lambda} = -\lambda$. Restriction of $\exp(Kt)$ on U is one-parametrer group of rotations. For the positive equilibrium P^* the intersection $(U + P^*) \cap \Delta_n$ is a convex polygon. It is forward invariant with respect to (1) because U is invariant, P^* is equilibrium and Δ_n is forward invariant. But a polygon on a plane cannot be invariant with respect to one-parametrer semigroup of rotations $\exp(Kt)$ $(t \geq 0)$. This contradiction proves the absence of purely imaginary eigenvalues.

[1] A Metzler matrix is a matrix with non-negative off-diagonal entries

The main result of this section is the following theorem:

Theorem 1. For every nonzero eigenvalue λ of matrix K

$$\frac{|\Im\lambda|}{|\Re\lambda|} \leq \cot\frac{\pi}{n}. \tag{3}$$

The the proof of this theorem can be extracted from the general Dmitriev–Dynkin–Karlelevich theorems (Dmitriev and Dynkin, 1946; Karpelevich, 1951). Due to space limitation, however, we do not provide it here and refer the reader to (Gorban et al., 2015).

Remark 1. It is important to note that the bound given in Theorem 1 is sharp. Indeed, let K define a directed ring with uniform weights q, e.g.

$$K = \begin{pmatrix} -q & 0 & 0 & \cdots & 0 & q \\ q & -q & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & & \ddots & q & -q & 0 \\ 0 & 0 & \cdots & 0 & q & -q \end{pmatrix}.$$

The eigenvalues of K are

$$\lambda_k = -q + q\exp\left(\frac{2\pi k \mathrm{i}}{n}\right), \quad k = 0, 1, \dots, n-1,$$

cf. (Davis, 1994), with $\mathrm{i} = \sqrt{-1}$ the imaginary unit. Thus $\frac{|\Im\lambda_1|}{|\Re\lambda_1|} = \cot\frac{\pi}{n}$. Note that for large n,

$$\cot\frac{\pi}{n} \approx \frac{n}{\pi},$$

which means that oscillations in a directed ring with a large number of systems decay very slowly.

An important consequence of this extremal property of a directed ring is that not only transients in the cycle decay very slowly but also that the overall behavior of transients becomes extremely sensitive to perturbations. This, as we show in the next sections, gives rise to resonances and bistabilities if neutrally stable nodes in (1) are replaced with the ones exhibiting oscillatory dynamics.

3. COUPLED NONLINEAR NEURAL OSCILLATORS

Consider now a network of FitzHugh-Nagumo (FHN) neurons

$$\begin{cases} \dot{z}_j = \alpha\,(y_j - \beta z_j) \\ \dot{y}_j = y_j - \gamma y_j^3 - z_j + u_j, \end{cases} \tag{4}$$

$j = 1, 2, \dots, n$ with parameters α, β, γ chosen as

$$\alpha = \tfrac{8}{100},\ \beta = \tfrac{8}{10},\ \gamma = \tfrac{1}{3}$$