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On practical synchronisation and collective behaviour of networked heterogeneous oscillators ⋆

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Abstract: We present preliminary results on synchronisation of nonlinear oscillators interconnected in heterogeneous networks that is, we assume that the systems' dynamic models are different, albeit of the same dimension. Under mild conditions, we show that the synchronisation errors may be diminished by increasing the interconnection gain. That is, we establish results on practical synchronisation. Although this problem has been studied in the literature, our approach is novel from an analytical perspective: the behaviour of the interconnected systems is determined by two main components, the stability of an averaged dynamics, relative to an attractor of what we call *emergent dynamics* and, secondly, the synchronisation of each individual oscillator relative to the emergent dynamics. Our framework is general, it covers as a particular case that of (set-point) consensus but also trajectory-tracking synchronisation and consensus over manifolds.

Keywords: Stability, synchronisation, networked systems, emergent dynamics.

1. INTRODUCTION

The collective behaviour of network-interconnected complex systems depends on some key factors, such as: the dynamics of the individual units, the interconnection among the nodes and the network structure. Network dynamics may be modelled via *ordinary* differential equations –*cf.* Pogromski and Nijmeijer (2001); Isidori et al. (2014),

$$\dot{\mathbf{x}}_i = f_i(\mathbf{x}_i) + B\mathbf{u}_i, \qquad i \in \mathcal{I} := \{1, \ldots, N\} \quad \text{(1a)}$$

$$\mathbf{y}_i = C\mathbf{x}_i, \quad \text{(1b)}$$

where $\mathbf{x}_i \in \mathbb{R}^n$, $\mathbf{u}_i \in \mathbb{R}^m$ and $\mathbf{y}_i \in \mathbb{R}^m$ denote the state, the input and the output of the ith unit, respectively. Usually, graph theory is employed to describe the topological (structural) properties of networks; a network of N nodes is defined by its $N \times N$ adjacency matrix $D = [d_{ij}]$ whose (i, j) element, denoted by d_{ij}, specifies an interconnection between the ith and jth nodes. From a dynamical systems point of view a general setting such as *e.g.*, in Blekhman et al. (1997); Nijmeijer and Rodríguez-Angeles (2003), synchronisation may be qualitatively measured by equating a functional of the trajectories to zero and measuring the distance of the latter to the synchronisation manifold. In the case of a network of identical nodes, *i.e.*, if $f_i = f_j$ for all $i, j \in \mathcal{I}$ this may be defined in the space of $\mathbf{x} := [\mathbf{x}_1^\top, \cdots, \mathbf{x}_N^\top]^\top$ as

$$\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^{nN} : \mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_N\} \quad (2)$$

Such stability problem may be approached in a number of ways, *e.g.*, using tools developed for semi-passive, incrementally passive or incrementally input-output stable systems –see Pogromski and Nijmeijer (2001); Pogromski et al. (1999); Jouffroy and Slotine (2004); Lohmiller and Slotine (2005); Scardovi et al. (2009); Franci et al. (2011). If the manifold $\mathcal{S}$ is stabilised one says that the networked units are synchronised.

In general, the nodes' interconnections depend on the strength of the coupling and on the nodes' state variables or on functions of the latter, *i.e.*, outputs which define the coupling terms. This may be nonlinear, as *e.g.*, in the case of the the well-known Kuramoto's oscillator –see Belykh et al. (2005); Corson et al. (2012). In this paper we consider a particular case of coupling which is known in the literature as *diffusive coupling*. We assume that all the units have inputs and outputs of the same dimension and that the coupling between the ith and jth units is defined as a weighted difference: $\gamma d_{ij}(\mathbf{y}_i - \mathbf{y}_j)$, where $\mathbf{y}_i$ and $\mathbf{y}_j$ are the outputs of the units i and j respectively, $d_{ij} > 0$ is constant and $\gamma > 0$ is the interconnection gain.

Depending on whether the nodes are identical or not the network is respectively called *homogeneous* or *heterogeneous*. The behaviour of networks of systems with non-identical models is more complex due to the fact that the synchronisation manifold $\mathcal{S}$ does not necessarily exist. An alternative approach based on stability theory, is to address the synchronisation problem in a *practical* sense that is, to admit that, asymptotically, the differences between the units' motions are bounded and become smaller for larger values of the interconnection gain, but they do not necessarily vanish. This is the approach that we pursue here.

For the purpose of analysis we propose to study the behaviour of network-interconnected systems via two separate properties: the stability of what we call the *emergent dynamics* and the synchronisation errors of each of the units' motions, relative to an averaged system, also called "mean-field" system. This formalism covers the classical paradigm of consensus of a collection of integrators, in which case the emergent dynamics is null and the mean field trajectory corresponds to a weighted average of the

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nodes' trajectories. Moreover, for a balanced graph, we know that all units reach consensus and the steady-state value is an equilibrium point corresponding to the average of the initial conditions –see Ren et al. (2007). In our framework, the emergent dynamics possesses a stable attractor, in contrast to (the particular case of) an equilibrium point. Then, we say that the network presents dynamic consensus if there exists an attractor $\mathcal{A}$, in the phase-space of the emergent state, such that the trajectories of all units are attracted to $\mathcal{A}$ asymptotically and remain close to it. In the setting of heterogeneous networks, only *practical* synchronisation is achievable in general that is, the trajectories of all units converge to a neighbourhood of the attractor of the emergent dynamics and remain close to this neighbourhood.

In section 2 we present the network model, suitable for analysis; in Section 4 we present our main statements, whose proofs are provided in Panteley (2015). In Section 5 we present some illustrative simulation results, before concluding with some remarks in SectionPsec:concl.

2. NETWORKS OF HETEROGENEOUS SYSTEMS

2.1 System model

We consider an undirected network composed of N heterogeneous diffusively coupled nonlinear dynamical systems in normal form:

$$\dot{\mathbf{y}}_i = f_i^1(\mathbf{y}_i, \mathbf{z}_i) + \mathbf{u}_i \tag{3a}$$

$$\dot{\mathbf{z}}_i = f_i^2(\mathbf{y}_i, \mathbf{z}_i) \tag{3b}$$

where $\mathbf{u}_i \in \mathbb{R}^m$ denote the inputs, $\mathbf{y}_i \in \mathbb{R}^m$ the outputs to be synchronised and the state $\mathbf{z}_i$ corresponds to that of the ith agent's zero-dynamics *i.e.*, $\dot{\mathbf{z}}_i = f_i^2(0, \mathbf{z}_i)$. The functions $f_i^1 : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$, $f_i^2 : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ are assumed to be locally Lipschitz. We consider that the network units are connected via *diffusive coupling*, *i.e.*, for the i-th unit the coupling is given by

$$\mathbf{u}_i = -\sigma \sum_{j=1}^{N} d_{ij}(\mathbf{y}_i - \mathbf{y}_j), \quad d_{ij} = d_{ji} \tag{4}$$

where $\sigma > 0$ corresponds to the coupling gain between the units and the individual interconnections weights, d_{ij} which define the symmetric Laplacian matrix,

$$L = \begin{bmatrix} \sum_{i=2}^{N} d_{1i} & -d_{12} & \dots & -d_{1N} \\ -d_{21} & \sum_{i=1, i\neq 2}^{N} d_{2i} & \dots & -d_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -d_{N1} & -d_{N2} & \dots & \sum_{i=1}^{N-1} d_{Ni}. \end{bmatrix}. \tag{5}$$

By construction, all row sums of L are equal to zero and all its eigenvalues of are real, exactly one of which (say, λ_1) equal to zero, while others are positive, *i.e.*, $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$– see Merris (1994).

Next, let $\mathbf{y} = [\mathbf{y}_1^\top \ \cdots \ \mathbf{y}_N^\top]^\top$, $\mathbf{u} = [\mathbf{u}_1^\top \ \cdots \ \mathbf{u}_N^\top]^\top$, $\mathbf{x} = [\mathbf{x}_1^\top \ \cdots \ \mathbf{x}_N^\top]^\top$, and define $F : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ as

$$F(\mathbf{x}) = \begin{bmatrix} F_1(\mathbf{x}_1) \\ \vdots \\ F_N(\mathbf{x}_N) \end{bmatrix}, \quad F_i(\mathbf{x}_i) = \begin{bmatrix} f_i^1(\mathbf{y}_i, \mathbf{z}_i) \\ f_i^2(\mathbf{y}_i, \mathbf{z}_i) \end{bmatrix}_{i \in \mathcal{I}}. \tag{6}$$

With this notation, the diffusive coupling inputs $\mathbf{u}_i$, defined in (4), can be re-written in the compact form

$$\mathbf{u} = -\sigma[L \otimes I_m]\mathbf{y},$$

where the symbol $\otimes$ stands for the right Kronecker product. Then, the network dynamics becomes

$$\dot{\mathbf{x}} = F(\mathbf{x}) - \sigma[L \otimes E_m]\mathbf{y} \tag{7a}$$

$$\mathbf{y} = [I_N \otimes E_m^\top]\mathbf{x}, \tag{7b}$$

where $E_m^\top = [I_m, 0_{m \times (n-m)}]$. The qualitative analysis of the solutions to the latter equations is our main subject of study.

2.2 Dynamic consensus and practical synchronisation

We generalise the consensus paradigm by introducing what we call *dynamic consensus*. This property is achieved by the systems interconnected over a network if and only if their motions converge to one generated by what we call *emergent dynamics*. In the case that the Laplacian is symmetric, the emergent dynamics is naturally defined as the average of the units' drifts that is, via the functions $f_s^1 : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$, $f_s^2 : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$, defined as

$$f_s^1 := \frac{1}{N}\sum_{i=1}^{N} f_i^1(\mathbf{y}_e, \mathbf{z}_e), \quad f_s^2 := \frac{1}{N}\sum_{i=1}^{N} f_i^2(\mathbf{y}_e, \mathbf{z}_e) \tag{8}$$

hence, the emergent dynamics may be written in the compact form

$$\dot{\mathbf{x}}_e = f_s(\mathbf{x}_e) \qquad \mathbf{x}_e = [\mathbf{y}_e^\top \ \mathbf{z}_e^\top]^\top, \quad f_s := [f_s^{1\top} \ f_s^{2\top}]^\top. \tag{9}$$

For the sake of comparison, in the classical (set-point) consensus paradigm, all systems achieving consensus converge to a common equilibrium *point* that is, $f_s \equiv 0$ and $\mathbf{x}_e$ is constant. In the case of formation tracking *control*, Equation (9) can be seen as the reference dynamics to the formation.

Next, we introduce the *average* state (also called mean-field) and its corresponding dynamics. Let

$$\mathbf{x}_s = \frac{1}{N}\sum_{i=1}^{N} \mathbf{x}_i \tag{10}$$

which comprises an average output, $\mathbf{y}_s \in \mathbb{R}^m$, defined as $\mathbf{y}_s = E_m^\top \mathbf{x}_s$ and the state of the average zero dynamics, $\mathbf{z}_s \in \mathbb{R}^{n-m}$, that is, $\mathbf{x}_s = [\mathbf{y}_s^\top, \mathbf{z}_s^\top]^\top$. Now, by differentiating on both sides of (10) and after a direct computation in which we use (3), (4) and the fact that the sums of the elements of the Laplacian's rows equal to zero, *i.e.*,

$$\frac{1}{N}\sum_{i=1}^{N} -\sigma\big[d_{i1}(\mathbf{y}_i - \mathbf{y}_1) + \cdots + d_{iN}(\mathbf{y}_i - \mathbf{y}_N)\big] = 0,$$

we obtain

$$\dot{\mathbf{y}}_s = \frac{1}{N}\sum_{i=1}^{N} f_i^1(\mathbf{y}_i, \mathbf{z}_i), \quad \dot{\mathbf{z}}_s = \frac{1}{N}\sum_{i=1}^{N} f_i^2(\mathbf{y}_i, \mathbf{z}_i).$$

Next, in order to write the latter in terms of the average state $\mathbf{x}_s$, we use the functions f_s^1 and f_s^2 defined above so, after (8) the *average dynamics* may be expressed as