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Optimal Control and Stochastic Synchronization of Phase Oscillators ⋆

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Abstract: Synchronization of oscillations is a phenomenon prevalent in natural, social, and engineering systems, such as neural circuitry in the brain, sleep cycles in biology, semiconductor lasers in physics, and periodic vibrations in mechanical engineering. The ability to control synchronization of oscillating systems then has important research and clinical implications, for example, for the study of brain functions. In this paper, we study optimal control and synchronization of nonlinear oscillators described by the phase model. We consider both deterministic and stochastic cases, in which we derive open-loop controls that create desired synchronization patterns and devise feedback controls that maximize the steady-state synchronization probability of coupled oscillators in the presence of shared and unshared noisy stimuli modeled by the Brownian motion.

Keywords: Stochastic control, nonlinear oscillators, synchronization, neurons, computational optimal control.

1. INTRODUCTION

Synchronization of oscillations is a phenomenon prevalent in natural, social, and engineering systems. The concept of synchronization is particularly significant to the study of biological systems (Strogatz [2000]), which exhibit endogenous oscillations with periods ranging from milliseconds, such as in spiking neurons (Izhikevich [2007]), to years, as in hibernation cycles (Mrosovsky [1980]), and important to physics, such as semiconductor lasers (Fischer et al. [2000]) and mechanical engineering, such as vibrating mechanical systems (Blekhman [1988]). Controlling synchronization then has compelling applications in a wide range from clinical medicine, such as protocols for coping with jet lag (Vosko et al. [2010]) and clinical treatments for neurological disorders including epilepsy and Parkinson's disease (L. Hofmann et. al [2011]), to the design of neurocomputers (Hoppensteadt and Izhikevich [1999]).

The dynamics of self-sustained oscillations in biological, physical, and engineering systems are often described in terms of limit cycle oscillators. The high-dimensional limit cycle dynamics can then be reduced to a representation with a single phase variable (Brown et al. [2004]). Such phase-reduced models, due to their simplicity, are very popular for modeling and analyzing dynamical properties of oscillators at both individual and population levels. For example, the minimum-energy waveform for entrainment of neuron oscillators to a desired forcing frequency and charge-balanced time-optimal and minimum-power controls for spiking a neuron at specified timing have been derived based on the phase model (Zlotnik and Li [2012], Dasanayake and Li [2014, 2011], Nabi and Moehlis [2010]). Synchronization engineering techniques, which utilized a global, nonlinear delayed feedback to induce a pre-selected synchronization structure, have been developed for effective control of the collective behavior of globally coupled nonlinear phase oscillators (Kiss et al. [2007]).

In practice, these oscillating systems, such as neurons, receive inputs that are inherently stochastic in nature due to, for example, random variation in inter-arrival times of presynaptic events. These external stimuli are well approximated by white noise when the neurons are constantly bombarded with many presynaptic inputs, and, moreover, involve both shared and unshared components, because, in addition to receiving global nonspecific background noise, some neurons respond to particular noisy stimuli but some do not (Ly and Ermentrout [2009]). Although phase models have been intensively employed to analyze synchronization of an ensemble of neuron oscillators, there exists little work on stochastic control of phase oscillators (Teramae and Tanaka [2004]), and the realization of a desired steady-state distribution in an ensemble of noisy oscillators using an external control has not been investigated.

In this paper, we design optimal waveforms to control synchronization of nonlinear oscillators described by the phase model. In particular, we consider the synchronization of two oscillators in the absence and presence of stochastic stimuli. In the next section, we introduce the phase model and derive open-loop controls that form specific synchronization patterns of two uncoupled oscillators. In Section

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3, we study the stochastic control of two coupled oscillators in the presence of additive noise modeled by Brownian motion. We find optimal feedback controls that maximize the steady-state synchronization probability of this system driven by correlated (shared) and independent (unshared) noise. Although in practice it is often desired to consider the control of a population of oscillators, theoretical developments based on a two-oscillator system is fundamental and insightful to the study of ensemble systems.

2. OPEN-LOOP CONTROL OF UNCOUPLED OSCILLATORS

2.1 Phase Models

The high-dimensional complex dynamics of an oscillating system described by an ordinary differential equation system $\dot{x} = F(x, v)$, where $x(t) \in \mathbb{R}^n$ is the state and $v(t) \in \mathbb{R}$ is a control, can be reduced by a model reduction technique to a single phase variable, given by

$$\dot{\theta} = f(\theta) + Z(\theta)u(t), \tag{1}$$

where $\theta \in \Theta = [0, 2\pi)$ is the phase variable, $f : \Theta \to \mathbb{R}$ represents the system's baseline dynamics, $Z : \Theta \to \mathbb{R}$ is known as the phase response curve (PRC), and $u(t) \in \mathbb{R}$ is the external stimulus. This model reduction is valid while the state, x, of the full dynamical system remains in a neighborhood of its unforced periodic orbit (Brown et al. [2004]), and hence phase models are accurate only when the control input u is weak. Phase models are widely employed in physics, chemistry, and biology (Pikovsky et al. [2003]) to study rhythmic systems where the oscillatory phase, but not the full state, can be observed, and where the PRC can be obtained experimentally.

2.2 Phase Pattern Formation of Two Oscillators

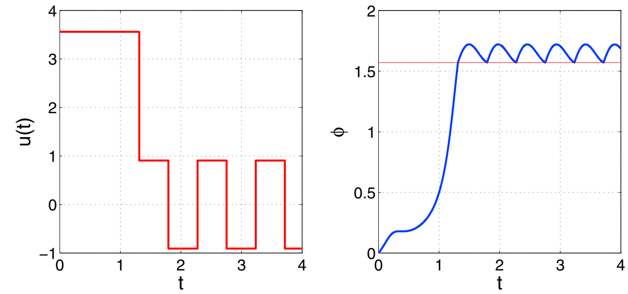
We consider a pair of oscillators described by the sinusoidal phase model, which receive a common control input. Let θ_i denote a representative oscillator from the ensemble i, $i = 1$ or 2. The dynamics of this two-oscillator system are given by

$$\dot{\theta}_1 = \omega_1 + u \sin\theta_1, \tag{2}$$

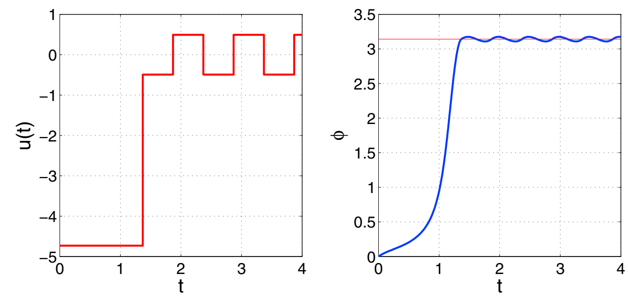
$$\dot{\theta}_2 = \omega_2 + u \sin\theta_2, \tag{3}$$

where we take $\omega_2 > \omega_1$. The common input $u(t)$ servers as a coupling of the two oscillators. We assume that a well-defined initial state can be established by a classical phase resetting method, where a strong pulse is applied to the oscillators and brings them to the same initial phase, independent of their previous phases (Winfree [1980]). We show that specific pattern formation between these two oscillators can be achieved by the application of piecewise constant open-loop controls. Note that our approach is different from that presented in (Hoppensteadt and Izhikevich [1999]), where the phase difference is built asymptotically in time.

Without loss of generality, we assume the common initial phase of the two oscillators $\theta_1(0) = \theta_2(0) = 0$. We consider the problem of designing controls to create a specific phase configuration of the two oscillators, i.e., $\theta_2(t) - \theta_1(t) = \alpha\pi$ for $\alpha \in (0, 1]$. Our control strategy is to first build the desired phase difference $\alpha\pi$ with a constant control and then design a sequence of piecewise constant controls to periodically maintain this configuration.



(a) Synchronization control and phase difference for $\alpha = \frac{1}{2}$



(b) Synchronization control and phase difference for $\alpha = 1$

Fig. 1. (a) The synchronization control (left panel) that builds and maintains $\pi/2$ phase difference between the two oscillators with $\omega_1 = 1.9\pi$ rad/sec and $\omega_2 = 2.1\pi$ rad/sec and the resulting phase difference trajectory (right panel) for $n = 1$ as in (5). The exact $\pi/2$ phase difference is maintained at every π phase evolution. (b) The synchronization control (left panel) that builds and maintains π phase difference and the resulting phase difference trajectory (right panel).

Let T_α be the time instance at which the two oscillators create the phase difference $\alpha\pi$ for some desired $\alpha \in (0, 1]$. A constant control input $u = U_\alpha$ that achieves this phase separation satisfies $\theta_1(T_\alpha) = 2n\pi$ and $\theta_2(T_\alpha) = (2n + \alpha)\pi$ for some positive integer n, which implies, by integrating (2) and (3),

$$\int_0^{2n\pi} \frac{d\theta_1}{\omega_1 + U_\alpha \sin\theta_1} = \int_0^{(2n+\alpha)\pi} \frac{d\theta_2}{\omega_2 + U_\alpha \sin\theta_2} = T_\alpha,$$

or, equivalently,

$$\frac{2n\pi}{\sqrt{\omega_1^2 - U_\alpha^2}} = \frac{2}{\sqrt{\omega_2^2 - U_\alpha^2}} \Bigg\{ n\pi + \tan^{-1}\Bigg[\frac{\omega_2}{\sqrt{\omega_2^2 - U_\alpha^2}} \cdot \left(\tan\frac{\alpha\pi}{2} + \frac{U_\alpha}{\omega_2} \right) \Bigg] - \tan^{-1} \frac{U_\alpha}{\sqrt{\omega_2^2 - U_\alpha^2}} \Bigg\} = T_\alpha. \tag{4}$$

Let $L(U_\alpha)$ and $R(U_\alpha)$ denote the left and the right hand side of the equality in (4), respectively. Then, we have $L(0) = 2n\pi/\omega_1$, $R(0) = (2n + \alpha)\pi/\omega_2$, $L(\omega_1) \to \infty$, and $R(\omega_1) < \infty$. Moreover, if

$$\frac{\omega_2}{\omega_1} < \frac{2n + \alpha}{2n}, \tag{5}$$

then $L(0) < R(0)$, while it is always $L(\omega_1) > R(\omega_1)$. Thus, when (5) is satisfied, the transcendental equation (4) has a solution $U_\alpha \in (0, \omega_1)$. The duration T_α corresponding to this U_α is given by $T_\alpha = L(U_\alpha) = \frac{2n\pi}{\sqrt{\omega_1^2 - U_\alpha^2}}$ from (4).