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Eigenvalue Placement Approach to Synchronization of Networked Nonlinear Oscillators

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Abstract: This paper is concerned with synchronization of nonlinear oscillators coupled via a directed network. It is known that that the synchronization can be achieved by choosing the coupling weights such that nonzero eigenvalues of the weighted Laplacian are located inside a certain convex region. We present a design method of the coupling weights which achieves the desired eigenvalue placement based on the bilinear matrix inequality optimization. The present method improves the previously reported method by the authors (Hibi and Takaba (2012)) by introducing a more general convex region for the eigenvalue placement.

Keywords: synchronization, regional eigenvalue placement, nonlinear oscillators

1. INTRODUCTION

Oscillatory phenomena are observed in many areas of natural sciences, and have been paid much attention from engineering fields for a long time.

A number of oscillating systems can be modeled as a system of many nonlinear oscillators mutually interacting through a network. In such a composite system, the oscillators often exhibit a common dynamic behavior. This phenomena is called *a synchronization*. Examples include circadian rhythms, heartbeats and fireflies flashing in unison (Barrat et al. (2008)).

Mathematical analysis of synchronization of coupled oscillators has been studied for years. For example, Pecora and Carroll (1998) derived a sufficient condition for achieving the synchronization under symmetric interactions, in terms of the eigenvalues of the Laplacian associated with the network. Nishikawa and Motter (2006a, 2006b) generalized the above condition to the case of asymmetric interactions. They also clarified the optimal network structure that maximizes the synchronizability.

The synchronization conditions derived in Pecora and Caroll (1998), Nishikawa and Motter (2006a, 2006b) give constraints of the location of the Laplacian eigenvalues into a certain region on the convex plane. Inspired by those works, the authors (Hibi and Takaba (2012)) considered the problem of designing coupling weights which synchronize nonlinear oscillators asymmetrically coupled over a network defined by a directed graph.Based on the regional pole placement technique (Chilali, Gahinet and Aplarian (1999)), the authors solved the probem by placing the eigenvalues of the weighted Laplacian into a prescribed convex subregion. Noted that, in Hibi and Takaba (2012), the convex subregion is formed as an intersection of disk regions centered on the real axis.

The purpose of this paper is to improve the authors' previous method mentioned above by introducing a more general convex subregion. To be more specific, we form the convex subregion in terms of disk regions cetered at arbitrary points on the complex plane, while the previous method requires the conters to lie on the real axis.

The remainder of this paper is organized as follows. In Section 2, we give a description of the coupled oscillators via graph theory and formulate the problems. Section 3 provides the algorithms for solving the problems numerically. In Section 4, a numerical example is presented in order to verify the effectiveness of the proposed method. The conclusion is given in Section 5.

Notations:

We use the following notations in this paper.
$\mathbb{R}_+$: non-negative real numbers
I_p : the $p \times p$ identity matrix
$\otimes$: Kronecker product
$\rho(\cdot)$: spectral radius of a matrix
$\mathrm{Re}(\cdot)$, $\mathrm{Im}(\cdot)$: the real and imaginary parts of a complex number
$\|\cdot\|$ denotes the Euclidean norm for a vector, and the maximum singular value for a matrix. The distance between a point x and a set S is defined by

$$\mathrm{dist}(x, S) = \inf_{y \in S} \|x - y\|.$$

The vector of ones $\mathbb{1}_p$ is defined by

$$\mathbb{1}_p := [1\ 1\ \cdots\ 1]^\top \in \mathbb{R}^p.$$

Furthermore, for Hermitian matrices A and B, the inequality $A - B < 0$ means that $A - B$ is negative definite.

2. PROBLEM FORMULATION

2.1 System Description

Throughout this paper, we consider a system of n identical oscillators coupled over a directed graph. Each oscillator, if isolated, evolves in accordance with the state equation

$$\frac{dx_i}{dt} = f(x_i), \quad i = 1, 2, \ldots, n, \tag{1}$$

where $f : \mathbb{R}^m \to \mathbb{R}^m$ is a C^1-function. We assume that this differential equation admits a limit cycle S_0 with period $T > 0$. Recall that a limit cycle is the image of a periodic solution projected onto the state-space $\mathbb{R}^m$.

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Interactions between the oscillators are performed through a network which is defined in terms of a directed graph. An example of such a graph is depicted in Fig. 1. We here introduce some basic notions from the graph theory (Godsil and Royle (2001)). A *directed graph* $\mathcal{G}$ is defined as a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the node set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. There is a directed edge from the node i to the node j if $(i, j) \in \mathcal{E}$. We denote the set of neighbors of the node $i \in \mathcal{V}$ by $\mathcal{N}_i := \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$.

The *weighted adjacency matrix* A consists of nonnegative weights on the edges, *i.e.*,

$$\begin{cases} a_{ij} \geq 0 & \text{if } (i, j) \in \mathcal{E}, \\ a_{ij} = 0 & \text{if } (i, j) \notin \mathcal{E}. \end{cases}$$

We sometimes refer to $(\mathcal{G}; A)$ as a *weighted directed graph*.

As shown in Fig. 1, each node represents each oscillator, and $(i, j) \in \mathcal{E}$ implies that the evolution of the oscillator i depends on the state of the oscillator j. Furthermore, the elements of A are identified with the coupling weights between oscillators. We say that the interactions between oscillators are *symmetric* if $(i, j) \in \mathcal{E}$ implies both $(j, i) \in \mathcal{E}$ and $a_{ij} = a_{ji}$. Otherwise, the interactions are said to be *asymmetric*.

The *weighted Laplacian* $L = [l_{ij}]_{n\times n}$ defined by the following equation will play a crucial role in this paper.

$$l_{ij} = \begin{cases} -a_{ij} & \text{if } i \neq j, \\ \sum_{j\neq i} a_{ij} & \text{if } i = j. \end{cases}$$

We will denote with $\mathcal{L}(\mathcal{G})$ the family of weighted Laplacian matrices associated with the digraph $\mathcal{G}$.

The most important feature of L is that it always has a zero eigenvalue with an eigenvector $\mathbb{1}_n$, namely $L\,\mathbb{1}_n = 0$. We thus denote the eigenvalues of L as

$$\underbrace{\lambda_1 = 0}_{n_1=1}, \underbrace{\lambda_2, \ldots, \lambda_2}_{n_2}, \underbrace{\lambda_3, \ldots, \lambda_3}_{n_3}, \cdots, \underbrace{\lambda_r, \ldots, \lambda_r}_{n_r},$$

where λ_1 is the aforementioned zero eigenvalue, the multiplicities n_k are the sizes of the Jordan blocks associated with λ_k, $k = 1, \ldots, r$, and $\sum_{k=1}^{r} n_k = n$. It is straightforward to verify that λ_1 is an isolated eigenvalue, *i.e.* $n_1 = 1$. Note that these eigenvalues can be complex numbers since a weighted Laplacian for a directed graph is not a symmetric matrix in general.

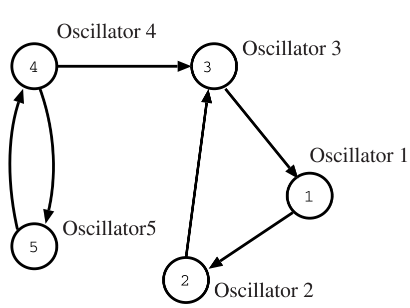

Fig. 1. Nonlinear oscillators coupled over a directed graph

When the oscillators are coupled over $\mathcal{G}$, the oscillator i interacts with its neighboring oscillators $j \in \mathcal{N}_i$, and its dynamics is expressed as

$$\frac{dx_i}{dt} = f(x_i) - \sum_{j=1}^{n} a_{ij}(h(x_i) - h(x_j)), \quad i = 1, 2, \ldots, n,.$$

where the function $h : \mathbb{R}^m \to \mathbb{R}^m$ is of class C^1. This equation can be rewritten as

$$\frac{dx_i}{dt} = f(x_i) - \sum_{j=1}^{n} l_{ij} h(x_j), \quad i = 1, 2, \ldots, n, \tag{2}$$

or in short

$$\frac{dx}{dt} = F(x) - (L \otimes I_m) H(x) \tag{3}$$

with

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad F : x \mapsto \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}, \quad H : x \mapsto \begin{bmatrix} h(x_1) \\ \vdots \\ h(x_n) \end{bmatrix}.$$

Synchronization of the coupled oscillators is characterized in terms of stability of the augmented limit cycle $\mathcal{S} := \{\, \mathbb{1}_n \otimes s \mid s \in S_0\}$ in $\mathbb{R}^{mn}$.

Definition 1. Synchronization is achieved for the coupled oscillators of (2) if the limit cycle $\mathcal{S}$ is asymptotically stable for (3).

Remark 1. A definition of stability of a limit cycle is given along the same line as Khalil (1996). For the nonlinear system (3), the limit cycle $\mathcal{S}$ is said to be *stable* if, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mathrm{dist}(x(0), \mathcal{S}) < \delta \Rightarrow \mathrm{dist}(x(t), \mathcal{S}) < \epsilon, \ {}^\forall t \in \mathbb{R}_+.$$

Furthermore, $\mathcal{S}$ is said to be *asymptotically stable* if it is stable and there exists $\delta > 0$ such that

$$\mathrm{dist}(x(0), \mathcal{S}) < \delta \Rightarrow \lim_{t\to\infty} \mathrm{dist}(x(t), \mathcal{S}) = 0.$$

2.2 A Sufficient Condition for Synchronization

The goal of this paper is to develop algorithms for designing coupling weights $\{a_{ij}\}_{i,j\in\mathcal{V}}$ which synchronize the coupled oscillators or synchronize them at a specified convergence speed. For this purpose, we first review the sufficient synchronization condition due to Nishikawa and Motter (2006a,2006b).

Recall that any element of $\mathcal{S}$ is given by $\mathbb{1} \otimes s$, where $s \in S_0$ is an arbitrary periodic solution of the oscillator dynamics (1). We also define

$$x_i(t) = s(t) + \xi_i(t), \quad i = 1, 2, \ldots, n.$$

Noting $ds/dt = f(s)$ and $\sum_{j=1}^{n} l_{ij} = 0$, we obtain the approximate linearization of (2) about $s(t)$ as

$$\frac{d\xi_i}{dt} = \mathbf{D}f(s(t))\xi_i - \sum_{j=1}^{n} l_{ij}\mathbf{D}h(s(t))\xi_j, \quad i = 1, \ldots, n, \tag{4}$$

where $\mathbf{D}f$ and $\mathbf{D}h$ are the Jacobian matrices of f and h, respectively, *i.e.*

$$\mathbf{D}f(s(t)) = \left.\frac{\partial f(x)}{\partial x}\right|_{x=s(t)}, \quad \mathbf{D}h(s(t)) = \left.\frac{\partial h(x)}{\partial x}\right|_{x=s(t)}.$$

We here make the following assumption.

Assumption 1. $\|\mathbf{D}h(\theta)\|$ is bounded with respect to $\theta \in \mathbb{R}^m$.