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Uniform hyperbolicity of locally maximal chaotic invariant sets in a system with only one stable equilibrium ⋆

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Abstract: By the methods of Poincaré map and the cone field theory, we present a rigorous computer-assisted method to verify the existence of a locally maximal uniformly hyperbolic chaotic invariant set for an autonomous chaotic system which has only one stable equilibrium. At the same time, our arguments show that the second return Poincaré map restricted to the hyperbolic set is indeed topologically conjugate to the 2-shift map.

Keywords: Chaos, uniformly hyperbolic set, cone field, Poincaré map.

1. INTRODUCTION

One of the most important topics in the modern theories of dynamical systems is the uniformly hyperbolic chaotic invariant sets (see Definition 2.1 below). The uniform hyperbolicity implies that all relevant trajectories in the invariant set are of chaotic saddle type with well defined stable and unstable directions. For discrete dynamical systems, as shown in Fisher (2006), there are many classical examples of uniformly hyperbolic invariant sets such as Smale horseshoe, Anosov torus automorphism and Solenoid attractor. All of the hyperbolic invariant sets of above examples possesses a significant property called local maximality (or isolation) (see Definition 2.2 below) with strong chaotic properties. Isolated hyperbolic invariant sets with chaotic properties possess many important properties, such as (see Wilczak (2010)): (a) structural stability: persist under any C^1 perturbation; (b) entropy and number of periodic points persist under perturbation; (c) periodic points are dense in the set of non-wandering points.

For flows or differential equations, some nonautonomous systems have been proved to have uniformly hyperbolic sets on which the dynamics are chaotic (see Kuznetsov (2005), Kuznetsov and Sataev (2007)), while, for autonomous systems, as far as we know, the hyperbolic invariant sets are seldom found. For example, the famous Lorenz attractor is not uniformly hyperbolic(see Tucker (2002)). Moreover, whether Lorenz attractor includes a locally maximal hyperbolic set with chaotic properties remains unknown. Therefore, it is interesting to find an simple autonomous chaotic system with a locally maximal hyperbolic chaotic invariant set.

Recently, a three-dimensional autonomous chaotic system with a unique stable node-focus equilibrium is reported in Wang and Chen (2012) as follows

$$\dot{x} = yz + 0.006,\ \dot{y} = x^2 - y,\ \dot{z} = 1 - 4x. \tag{1}$$

For this striking chaotic system, despite the well-known Si'lnikov criterions are not applicable, Wang and Chen still numerically discovered the chaotic behavior. Later, a rigorous computer-assisted verification of horseshoe chaos is given in Huan et al. (2013) by virtue of topological horseshoe theory(see Yang (2004) and Yang (2009)). Huan et al. obtained that, for a cross-section, the second return Poincaré map of system (1) possesses a closed invariant set, on which it is topologically semi-conjugate to the 2-shift map. Nevertheless, one has no information on the uniform hyperbolicity of system (1) so far.

In the present paper, using Poincaré map and the cone field theory, we present a rigorous computer-assited verification to get a locally maximal hyperbolic set for the second return Poincaré map derived from system (1). Furthermore, without using the theory of topological horseshoe, we obtain a stronger result than the result obtained in Huan et al. (2013), i.e., the second return Poincaré map restricted to the locally maximal hyperbolic set is in fact *topologically conjugate* to 2-shift map.

2. THE BASIC DEFINITIONS, RESULTS AND SOME EXTENSIONS

We first review some definitions and results related to hyperbolicity.

Let f be a diffeomorphism on a manifold M and $\Lambda \subset M$

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be a compact invariant set of f. We denote by $T\Lambda$ the restriction of the tangent bundle TM to Λ.

Definition 2.1(Wiggins (2003)) We say f is *uniformly hyperbolic* on Λ, or Λ is a *uniformly hyperbolic invariant set* of f, if the following assumptions hold:

• $T\Lambda$ can split into a direct sum $T\Lambda = E^s \oplus E^u$. E^s and E^u are $Tf-$invariant subbundles, i.e,

$$Df(x)E^s(x) = E^s(f(x)), \forall x \in \Lambda,$$
$$Df(x)E^u(x) = E^u(f(x)), \forall x \in \Lambda.$$

• There are constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|Df^n(x)v\| < C\lambda^n\|v\|, \text{ for } \forall x \in \Lambda,\ v \in E^s_x,\ n \geq 0,$$
$$\|Df^{-n}(x)v\| < C\lambda^n\|v\|, \text{ for } \forall x \in \Lambda,\ v \in E^u_x,\ n \geq 0.$$

Here $\|\cdot\|$ denotes a metric on M.

Definition 2.2(Crovisier (2001)) A invariant set Λ is said to be *locally maximal*(or *isolated*) if there exists a open neighborhood V in M such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(V)$ and V is called an *isolating neighborhood* of Λ.

Remark. Naturally, a set Λ is said to be *locally maximal (or isolated) hyperbolic invariant set* if it is uniformly hyperbolic invariant and locally maximal.

Proposition 1 *Suppose that A is a compact subset of M, if $A' = \bigcap_{n=-\infty}^{\infty} f^n(A) \neq \emptyset$, then A' is an invariant set of f. Moreover, if there exist $l, k \in \mathbb{Z}, l \geq k$ such that $\bigcap_{n=l}^{k} f^n(A) \subset \mathrm{int}(A)$, then A' is a locally maximal invariant set of f with $\mathrm{int}(A)$ as one of the isolating neighborhoods of A'. Here $\mathrm{int}(A)$ means the interior of A.*

Proof. The proof is trivial, we omit it. ■

Now, we give an introduction to sufficient conditions for a two-dimensional diffeomorphism to have a hyperbolic invariant set on which the dynamics are topologically conjugate to the 2-shift map, which is the important theoretical tools for the computer-assisted verification in this paper. These conditions were first given by Moser (1973), and then improved by Wiggins (2003).

Let α be a plane vector. For $a, b \in \mathbb{R}^2, a \neq b$, we denote the absolute value of the tangent value of the angle between α and $\overrightarrow{ab}$ by μ^α_{ab}, i.e., $\mu^\alpha_{ab} = \frac{1}{|\langle\overrightarrow{ab},\alpha\rangle|}(|\overrightarrow{ab}|^2|\alpha|^2 - |\langle\overrightarrow{ab},\alpha\rangle|^2)^{\frac{1}{2}}$, where $\langle,\rangle$ means the inner product of vectors.

Definition 2.3 A *μ-curve along vector α* is a bounded continuous curve L for which

$$\mu^\alpha_{ab} \leq \mu, \text{ for } \forall a, b \in L, a \neq b.$$

Remark. In fact, the constant μ can be interpreted as a bound on the slope of the curve L with respect to α.

As shown in Fig. 1, let M be a compact connected region of $\mathbb{R}^2$, plane sector $s \perp u$. $H_i = |h^1_i h^2_i h^3_i h^4_i|$ and $D_i = |d^1_i d^2_i d^3_i d^4_i|$ $(i = 1, 2, ..., N)$ are compact connected curved quadrilateral subsets of M satisfying the following three conditions: a) for $1 \leq i, j \leq N$, $i \neq j$, $H_i \cap H_j = \emptyset$, $D_i \cap D_j = \emptyset$; b) for $i = 1, 2, ..., N$, the boundaries $|h^1_i h^2_i|$

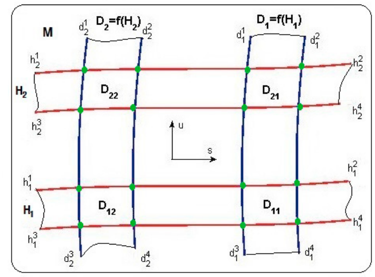

Fig. 1. $N = 2$ for illustrative purposes,the mapping relations of H_i and D_j for $1 \leq i, j \leq 2$.

and $|h^3_i h^4_i|$ (red lines in Fig. 1) are both μ-curve along the sector s and the boundaries $|d^1_i d^3_i|$ and $|d^2_i d^4_i|$ (blue lines in Fig. 1) are both λ-curve along the sector u; c) for $1 \leq i, j \leq N$, each μ-curve along the sector s in H_i mutually intersects with each λ-curve along the sector u in D_j (green intersections in Fig. 1) .

Throughout the rest of this section, we consider a diffeomorphism $f : M \to \mathbb{R}^2$. Suppose that f satisfies the following Assumption 1.

Assumption 1.

• *$0 \leq \mu\lambda < 1$, f maps H_i homeomorphically onto D_i, $(f(H_i) = D_i)$ for $i = 1, 2, ..., N$.*

• *For $i = 1, 2, ..., N$, the four boundaries of H_i respectively map to the four boundaries of D_i with the following principle: the μ-curves along the sector s (red lines in Fig. 1) do not map to the λ-curves along the sector u (blue lines in Fig. 1).*

We define $f(H_i) \cap H_j = D_{ji}$ and $H_i \cap f^{-1}(H_j) \equiv H_{ij} = f^{-1}(D_{ji})$, for $1 \leq i, j \leq N$. We further define $\mathcal{H} = \bigcup_{1\leq i,j\leq N} H_{ij}$ and $\mathcal{D} = \bigcup_{1\leq i,j\leq N} D_{ij}$. It should be obvious that $f(\mathcal{H}) = \mathcal{D}$.

Definition 2.4 For any point $x_0 \in \mathcal{H} \cup \mathcal{D}$, we define the *$\mu$-cone field along s at x_0* as follows

$$\mathcal{S}^s_{x_0} = \{z \in \mathbb{R}^2 \,|\, |z|_u \leq \mu|z|_s\},$$

where $|z|_u$ and $|z|_s$ represent the length of the projection of z along sector u and s, respectively, i.e., $|z|_u = \frac{|\langle z,u\rangle|}{|u|}$, $|z|_s = \frac{|\langle z,s\rangle|}{|s|}$. Similarly, the *$\lambda$-cone field along u at x_0* is defined as

$$\mathcal{S}^u_{x_0} = \{z \in \mathbb{R}^2 \,|\, |z|_s \leq \lambda|z|_u\}.$$

Then, we define the sector bundles

$$\mathcal{S}^s_{\mathcal{H}} = \bigcup_{x_0\in\mathcal{H}} \mathcal{S}^s_{x_0},\ \mathcal{S}^s_{\mathcal{D}} = \bigcup_{y_0\in\mathcal{D}} \mathcal{S}^s_{y_0},\ \mathcal{S}^u_{\mathcal{H}} = \bigcup_{x_0\in\mathcal{H}} \mathcal{S}^u_{x_0},$$
$$\mathcal{S}^u_{\mathcal{D}} = \bigcup_{y_0\in\mathcal{D}} \mathcal{S}^u_{y_0}.$$

Also, suppose f satisfies the following Assumption 2.

Assumption 2 (The cone field conditions). $Df(\mathcal{S}^u_{\mathcal{H}}) \subset \mathcal{S}^u_{\mathcal{D}}$, $Df^{-1}(\mathcal{S}^s_{\mathcal{D}}) \subset \mathcal{S}^s_{\mathcal{H}}$. Moreover

$$|Df(x_0)z|_u \geq \frac{1}{\epsilon}|z|_u, \forall x_0 \in \mathcal{H}, z \in \mathcal{S}^u_{x_0},$$
$$|Df^{-1}(y_0)z|_s \geq \frac{1}{\epsilon}|z|_s, \forall y_0 \in \mathcal{D}, z \in \mathcal{S}^s_{y_0},$$