Proceedings of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

On Networked Evolutionary Games Part 2: Dynamics and Control ⋆

Daizhan Cheng * **Fehuang He** ** **Tingting Xu** *

* *Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, P.R.China (e-mail: dcheng@iss.ac.cn, xutingtinghello@126.com)*
** *Institute of Astronautics, Harbin Institute of Technology, Harbin 150080, P.R.China (e-mail: hefenghua@gmail.com)*

Abstract: As Part 2 of the paper "on networked evolutionary games", this paper uses the framework presented in Part 1 (Qi et al., 2014) to explore further issues about networked evolutionary games (NEGs). First, the strategy profile dynamics (SPD) is constructed from the fundamental evolutionary equations (FEEs). Using SPD, the control of NEGs are investigated. Detailed mathematical models are obtained for both deterministic and dynamic cases respectively. Then certain more complicated NEGs are explored. They are: (i) NEG with strategies of different length information, which allows some players use longer history information such as the information at t and $t-1$ or so; (ii) NEG with Multi-Species, which allows an NEG with various kinds of players, they play several different fundamental network games according to their identities. (iii) NEG with time-varying payoffs. Since payoffs determine the evolution, the network profile dynamics will be a time-varying one. These more complicated NEGs can cover more general evolutions and they generalized the method proposed in Cheng et al. (Preprint2013).

Keywords: Networked evolutionary game, fundamental evolutionary equation, network profile dynamics, heterogeneous NEG, semi-tensor product of matrices

1. INTRODUCTION

In Part 1 of this paper an NEG is defined as following, which was firstly proposed in Cheng et al. (Preprint2013).

Definition 1. An NEG, game, denoted by $((N,E),G,\Pi)$, consists of three ingredients as:

(i) a network (graph) (N,E);
(ii) a fundamental network game (FNG), G, such that if $(i,j)\in E$, then i and j play the FNG with strategies $x_i(t)$ and $x_j(t)$ respectively.
(iii) a local information based strategy updating rule (SUR).

It was proved that the fundamental evolutionary equation (FEE) for each player can be obtained as

$$x_i(t+1)=f_i\left(\{x_k(t)|k\in U_2(i)\}\right),\quad i=1,\cdots,n. \tag{1}$$

Then the network profile dynamics is uniquely determined by FEEs.

We refer to Qi et al. (2014) and Cheng et al. (Preprint2013) for details.

Part 2 of the paper considers several advanced problems about NEGs. In Section 2 the SPD is constructed from FEEs. Using SPD, the control problems of NEGs are investigated. A detailed mathematical framework is presented in Section 3 as a standard k-valued logical control networks. Then all the techniques for the control of k-valued logical networks can be used. Section 4 considers the NEGs where players can use different length of historical information to update their strategies. In Section 5 we consider the NEGs with multi-species. That is, the players are classified into several species, and players of different species play different roles in the networked games. Section 6 considers when the fundamental network game has time-varying payoff functions. Section 7 is a brief conclusion.

2. FROM FEE TO NPD

The NPD is used to describe the evolution of the overall networked games. This section consider how to construct the NPD of an NEG using its nodes' FEEs. We consider two cases: (i) the FEEs are deterministic model; (ii) the FEEs are probabilistic model.

2.1 Deterministic Model

Assume

$$\begin{cases}x_1(t+1)=M_1x(t),\\ \vdots\\ x_n(t+1)=M_nx(t),\end{cases} \tag{2}$$

⋆ This work was supported in part by National Natural Science Foundation (NNSF) of China under Grants 61074114, 61273013, and 61333001.

where $x(t) = \ltimes_{i=1}^n x_i(t)$ and $M_i \in \mathcal{L}_{k\times k^n}$. Then we have the NPD as

$$x(t+1) = Mx(t), \tag{3}$$

where

$$M = M_1 * M_2 * \cdots * M_n \in \mathcal{L}_{k^n\times k^n}. \tag{4}$$

Example 2. Recall Example 12 of Part 1. We have

$$\begin{aligned} x_1(t+1) &= M_f x_4(t)x_5(t)x_1(t)x_2(t)x_3(t) \\ &= M_f W_{[2^3,2^2]}x(t) := M_1x(t), \\ x_2(t+1) &= M_f x_5(t)x_1(t)x_2(t)x_3(t)x_4(t) \\ &= M_f W_{[2^4,2]}x(t) := M_2x(t), \\ x_3(t+1) &= M_f x(t) := M_3x(t), \\ x_4(t+1) &= M_f x_2(t)x_3(t)x_4(t)x_5(t)x_1(t) \\ &= M_f W_{[2,2^4]}x(t) := M_4x(t), \\ x_5(t+1) &= M_f x_3(t)x_4(t)x_5(t)x_1(t)x_2(t) \\ &= M_f W_{[2^2,2^3]}x(t) := M_5x(t). \end{aligned}$$

Finally, we have the NPD as

$$x(t+1) = Mx(t), \tag{5}$$

where

$$\begin{aligned} M &= M_1 * M_2 * M_3 * M_4 * M_5 \\ &= \delta_{32}[1,20,8,24,15,32,16,32,29,32,32,32,31,32,32,32 \\ &\qquad 26,28,32,32,32,32,32,32,30,32,32,32,32,32,32,32]. \end{aligned} \tag{6}$$

2.2 Probabilistic Model

Assume the strategies have the probabilistic k-valued logical form as

$$x_i(t+1) = M_1^j x(t), \quad \text{with} \quad Pr = p_i^j, \\ j = 1,\cdots,s_i;\ i = 1,\cdots,n. \tag{7}$$

Then we have

$$x(t+1) = Mx(t), \tag{8}$$

where $M \in \Upsilon_{k^n\times k^n}$ can be calculated as

$$M = \sum_{j_1=1}^{s_1}\sum_{j_2=1}^{s_2}\cdots\sum_{j_n=1}^{s_n}\left[\left(\prod_{i=1}^n p_i^{j_i}\right) M_1^{j_1} * M_2^{j_2} * \cdots * M_n^{j_n}\right]. \tag{9}$$

We use an example to depict it.

Example 3. Recall Example 13 of Part 1. In fact, we can use Table 5 there to calculate M row by row. For instance, it is obvious that

$$\mathrm{Col}_1(M) = \mathrm{Col}_2(M) = \mathrm{Col}_3(M) = \delta_{32}^1.$$

As for $\mathrm{Col}_4(M)$, with probability 1/4 it could be δ_{32}^3 or δ_{32}^4 or δ_{32}^7 or δ_{32}^8. That is,

$$\begin{aligned}\mathrm{Col}_4(M) = [&0,0,\tfrac14,\tfrac14,0,0,\tfrac14,\tfrac14,0,0,0,0,0,0,0,0,\\ &0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]^T.\end{aligned}$$

We simply express it as

$$\delta_{32}\left[3/\tfrac14 + 4/\tfrac14 + 7/\tfrac14 + 8/\tfrac14\right].$$

Using this notation and a similar computation, we have

$$\begin{aligned} M = \delta_{32}[&1,1,1,\alpha,1,\alpha,\beta,\gamma,\mu,\lambda,11,32,\lambda,32,32,32 \\ &1,1,1,\alpha,1,22,\alpha,p,\mu,q,r,32,s,32,32,32], \end{aligned} \tag{10}$$

where

$$\begin{aligned} \alpha &= 3/\tfrac14 + 4/\tfrac14 + 7/\tfrac14 + 8/\tfrac14, \\ \beta &= 3/\tfrac12 + 7/\tfrac12, \\ \gamma &= 8/\tfrac13 + 16/\tfrac23, \\ \mu &= 1/\tfrac23 + 9/\tfrac13, \\ \lambda &= 18/\tfrac16 + 20/\tfrac16 + 26/\tfrac13 + 28/\tfrac13, \\ p &= 24/\tfrac13 + 32/\tfrac23, \\ q &= 26/\tfrac12 + 28/\tfrac12, \\ r &= 27/\tfrac14 + 28/\tfrac14 + 31/\tfrac14 + 32/\tfrac14, \\ s &= 29/\tfrac12 + 31/\tfrac12. \end{aligned}$$

3. MODELING CONTROLLED NEGS

Definition 4. Let $((N,E),G,\Pi)$ be an NEG, and $N = U \cup Z$ be a partition of N. We call $((U\cup Z),E),G,\Pi)$ a controlled NEG, if the strategies of $u \in U$ can be chosen arbitrarily. As a result, $z \in Z$ is called a state and $u \in U$ is called a control.

Using FEE, the strategy evolutionary equations can be expressed as (Cheng et al., Preprint2013)

$$x_i(t+1) = M_i x(t), \quad i = 1,\cdots,n, \tag{11}$$

where $x(t) = \ltimes_{j=1}^n x_j(t)$. Assume $U = \{i_1,\cdots,i_q\}$ with $1 \le i_1 < i_2 < \cdots < i_q \le n$, and $Z = \{j_1,j_2,\cdots,j_p\}$ with $1 \le j_1 < j_2 < \cdots < j_p \le n$, where $p+q = n$. Define $u_r = x_{i_r}$, $r = 1,\cdots,q$, and $z_s = x_{j_s}$, $s = 1,\cdots,p$.

We consider the deterministic case and the probabilistic case separately.

(1) (Deterministic Case) Assume $M_i \in \mathcal{L}_{k\times k^n}$. Then we have

$$\begin{aligned} z_s(t+1) &= x_{j_s}(t+1) = M_{j_s} \ltimes_{i=1}^n x_i(t) \\ &= M_{j_s} W_{[k,k^{i_q-1}]} u_q(t) x_1(t) \ltimes x_2(t) \ltimes \cdots \\ &\quad \hat{x}_{i_q} \ltimes \cdots \ltimes x_n(t) \\ &= M_{j_s} W_{[k,k^{i_q-1}]} W_{[k,k^{i_{q-1}}]} u_{q-1}(t) u_q(t) \\ &\quad x_1(t) \ltimes x_2(t) \ltimes \cdots \ltimes \hat{x}_{i_{q-1}} \ltimes \cdots \\ &\quad \hat{x}_{i_q} \ltimes \cdots \ltimes x_n(t) \\ &= \cdots \\ &= M_{j_s} \ltimes_{r=m}^1 W_{[k,k^{i_r+m-r-1}]} u(t) z(t), \end{aligned}$$

where $u(t) = \ltimes_{i=1}^q u_i(t)$, and $z(t) = \ltimes_{i=1}^p z_i(t)$. The notation $\hat{x}_s$ means this factor is removed.

Define

$$\Psi_s := M_{j_s} \ltimes_{r=m}^1 W_{[k,k^{i_r+m-r-1}]} \in \mathcal{L}_{k\times k^n}, \tag{12}$$

then we have

$$z_s(t+1) = \Psi_s u(t) z(t), \quad s = 1,\cdots,p. \tag{13}$$

Set

$$\Psi := \Psi_1 * \Psi_2 * \cdots * \Psi_p \in \mathcal{L}_{k^p\times k^n}. \tag{14}$$

The controlled network profile evolutionary equation is expressed as

$$z(t+1) = \Psi u(t) z(t). \tag{15}$$

This is a standard k-valued logical control network.

(2) (Probabilistic Case) Assume

$$M_i = M_i^{j_i} \in \mathcal{L}_{k\times k^n}, \text{ with } Pr = p_i^{j_i} \\ j = 1,\cdots,r_i,\ i = 1,\cdots,n. \tag{16}$$

Then for each choice: $\{j_1,\cdots,j_n \mid 1 \le j_i \le r_i\}$ we can use $\{M_i^{j_i} \mid i = 1,\cdots,n\}$ to construct $\Psi^{j_1,\cdots,j_n}$,