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Reduced equations of motion for a wheeled inverted pendulum

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Abstract: This paper develops the equations of motion in the reduced space for the wheeled inverted pendulum, which is an underactuated mechanical system subject to nonholonomic constraints. The equations are derived from the Lagrange-d'Alembert principle using variations consistent with the constraints. The equations are first derived in the shape space, and then, a coordinate transformation is performed to get the equations of motion in more suitable coordinates for the purpose of control.

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1. INTRODUCTION

The Wheeled Inverted Pendulum (WIP) - and its commercial version, the Segway - has gained interest in the past several years due to its maneuverability and simple construction (see e.g. Grasser et al. [2002], Segway [2015, Jan]). Other robotic systems based on the WIP are becoming popular as well in the robotic community for human assistance or transportation as can be seen in the works of Li et al. [2012], Nasrallah et al. [2007], Baloh and Parent [2003]. A WIP consists of a vertical body with two coaxial driven wheels.

The stabilization and tracking control for the WIP is challenging: the system belongs to the class of underactuated mechanical systems, since the control inputs are less than the number of configuration variables: There are a total of two control variables τ_1 and τ_2 which are the torques applied to rotate the wheels, and six configuration variables, namely, the x- and y- position of the WIP on the horizontal plane, the relative rotation angle of each of the wheels with respect to the body ϕ_1 and ϕ_2, the orientation angle θ, and the tilting angle α. In addition, the system is restricted by nonholonomic (nonintegrable) constraints and is thus not smoothly stabilizable at a point as proven by Brockett [1983]. These constraints do not restrict the state space on which the dynamics evolve, but the motion direction at a given point: The rolling constraint impedes a sideways motion, and the forward velocity of the WIP and its yaw rate are directly given by the angular velocity of the wheels. Wheeled robots have largely been considered as purely kinematic systems, due to the simplification in the motion and controllability analysis. The WIP, however, needs to be stabilized by dynamic effects, such that the complete dynamics need to be taken into account. In mechanical systems with nonholonomic constraints the configuration space Q is a finite dimensional smooth manifold, TQ is the tangent bundle - the velocity phase space - and a smooth (non-integrable) distribution $\mathcal{D} \subset TQ$ defines the constraints [1]. While traditional approaches like the Lagrange-d'Alembert equations lead to the equations of motion of nonholonomic mechanical systems (see, e. g., Pathak et al. [2005]), geometric approaches help to understand the structure and the intrinsic properties of the system. There is a lot of work regarding the modeling of nonholonomic systems, see for example Bloch [2003], Ostrowski [1999], Bloch et al. [1996], Bloch et al. [2009] and the references therein. These geometric tools help understand the mechanism of locomotion, i. e., the way motion is generated by changing the shape of the mechanical system.

Symmetries can be exploited to develop dynamical models in a reduced space. Roughly speaking, the Lagrangian L exhibits a symmetry if it does not depend on one configuration variable, lets say, q_j. The variable q_j is called cyclic. The Lagrangian is thus invariant under transformations in cyclic coordinates. Lie group action and symmetry reduction has been successfully applied to model other types of nonholonomic mechanical systems in the differential geometric framework. See for example the works by Bloch et al. [1996], Ostrowski [1999], Gajbhiye and Banavar [2012]. As shown, e. g., by Ostrowski [1999], the resulting equations can be put in a simplified form containing apart from the reduced equations of motion, also the momentum and reconstruction equation, which describe the dynamics of the system along the group directions. That is, how the system translates and rotates in space due to the change in the shape variables. Bloch et al. [2009] further show the advantage of using the Hamel equations to obtain the reduced nonholonomic equations of motion: The momentum equation is in this case given in a body frame which appears to be more natural than in a spatial frame, for the latter is rarely conserved for systems with nonholonomic constraints. The derivation of the reduced nonholomonic equations can be done as well using the constrained Lagrangian and a so-called *Ehresmann connection* which relates motion along the shape directions with the motion

[1] The distribution $\mathcal{D}$ defines the admissible velocities

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along the group directions. The approach is based on taking admissible virtual displacements from the Lagrange d'Alembert principle. Admissible means, that the variations satisfy the constraints (given by the connection). This paper follows this modeling tool. Note that we are not imposing the constraints before taking variations, we are taking variations according to the constraints.

Several control laws have been applied to the WIP, mostly using linearized models as can be seen in Li et al. [2012]. There is still the need to exploit the nonlinear geometric structure of the WIP to stabilize and control the system using coordinate-free control laws. Nasrallah et al. [2007] develop a model based on the Euler-Rodrigues parameters and analyze the controllability of the WIP moving on an inclined plane. Pathak et al. [2005] develop a model using the Lagrange-d'Alembert equations and check the strong accessibility condition. The aim of this note is to explore the motion of the system in the reduced (shape) space which lead to some net displacement of the mobile robot (motion in the group space) independently from the starting point. Additionally, we present the equations of motion in more suitable coordinates [2] for control or trajectory planning purposes: Since the shape space of the WIP is not fully actuated, the control task becomes difficult in these coordinates. The choice of the model can be done depending on which better suits the task.

Notation: Contrary to most of the literature, we use here the matrix/vector representation instead of the index convention. Readers are encouraged to read the referred literature for the wide-used index convention. Further, we use the following simplified notation for the transposed Jacobian: $\partial_x^T = \left(\frac{\partial}{\partial x}\right)^T$.

2. EQUATIONS OF MOTION IN SHAPE SPACE

Consider the configuration space $Q = G \times S$, where S denotes the *shape* space and G denotes the *group* space: Q is a trivial principal bundle with fibers G over a base manifold S. The shape space, as the name suggests, denotes the space of the possible shapes of the system. As stated by Ostrowski [1999], this division is natural in mechanisms that locomote, like mobile robots, where position changes are generated by (mostly cyclic) changes in the shape. See for example the oscillations of a snake-board which create the forward motion, or the rotation of the wheels of a mobile platform resulting in a platforms displacement due to the rolling-without-slipping interaction with the environment. The internal shapes of the WIP are solely defined by the relative angles of the wheels with respect to the body. And since the gravity acts on the WIP depending on the tilting angle (the gravity breaks the symmetry), and it is crucial for the stability of the system, the tilting angle is also considered as a shape variable (more on that later). Note that the net motion resulting from a change in shape is independent from the initial position (we assume, that the WIP is moving on a horizontal plane). Mathematically speaking is this nothing but an invariance (symmetry) of the Lagrangian under a change in position (group) coordinates. We are therefore interested in the reduced equations of motion in shape space variables.

[2] The same equations of motion can be found in Pathak et al. [2005]

On the configuration space $Q = G \times S$, the Lagrangian is a function $L : TG \times TS \to \mathbb{R}$ and the distribution characterizing the nonholonomic constraints is given by $\mathcal{D} \subset TQ$. A curve $q(t)$ on Q is said to satisfy the constraints if $\dot{q}(t) \in \mathcal{D}_q, \forall t$. This nonholonomic restriction can also be given in local coordinates as

$$\dot{g} + \mathbb{A}^T \dot{s} = 0, \tag{1}$$

where $g \in G$ and $s \in S$, and the matrix $\mathbb{A}$ describes how $\dot{g}$ and $\dot{s}$ are related to each other due to the constraints. Recall that the equations governing the dynamics of the system satisfy the Lagrange-d'Alembert Principle (F_{ext} denote the external forces)

$$\delta \int L(q, \dot{q}) dt + \int F_{ext}^T \, \delta q \, dt = 0, \tag{2}$$

which is equivalent to

$$\int \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \left(\frac{\partial L}{\partial q} \right) - F_{ext}^T \right] \delta q \, dt = 0. \tag{3}$$

Independent from the Lie-group structure, we can intrinsically eliminate the Lagrange-multipliers which arise from the constraint forces and write the reduced equations of motion using the Ehresmann connection (1), which is nothing but a way to split the tangent space into a horizontal (tangent to the shape space) and a vertical (tangent to the group space) part [3]. The curves $q(t)$ solving the equations of motion need to satisfy the constraints. Thus, the variations $\delta q = (\delta s, \delta g)$ are of the form $\delta g + \mathbb{A}^T \delta s = 0$ (see Bloch et al. [1996]). We assume, that the external forces are input torques τ and only act on the shape variables, i. e., $F_{ext}^T \, \delta q = \tau^T \, \delta s$. This assumption is valid, since we will consider group space motion only as a result of a change in the shape variables, and we neglect friction forces. Equation (3) takes the following form

$$\begin{aligned} &\int \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{s}} \right) - \left(\frac{\partial L}{\partial s} \right) - \tau^T \right] \delta s \, dt \\ &- \int \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{g}} \right) - \left(\frac{\partial L}{\partial g} \right) \right] \mathbb{A}^T \delta s \, dt = 0. \end{aligned} \tag{4}$$

To eliminate the group velocities $\dot{g}$, define the constrained Lagrangian

$$L_c(s, g, \dot{s}) = L(s, g, \dot{s}, -\mathbb{A}^T \dot{s}). \tag{5}$$

The following relationships hold

$$\frac{\partial L_c}{\partial \dot{s}} = \frac{\partial L}{\partial \dot{s}} + \frac{\partial L}{\partial \dot{g}} \frac{\partial \dot{g}}{\partial \dot{s}} = \frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial \dot{g}} \mathbb{A}^T \tag{6}$$

$$\frac{\partial L_c}{\partial s} = \frac{\partial L}{\partial s} + \frac{\partial L}{\partial \dot{g}} \frac{\partial \dot{g}}{\partial s} = \frac{\partial L}{\partial s} - \frac{\partial L}{\partial \dot{g}} \frac{\partial (\mathbb{A}^T \dot{s})}{\partial s} \tag{7}$$

$$\frac{\partial L_c}{\partial g} = \frac{\partial L}{\partial g} + \frac{\partial L}{\partial \dot{g}} \frac{\partial \dot{g}}{\partial g} = \frac{\partial L}{\partial g} - \frac{\partial L}{\partial \dot{g}} \frac{\partial (\mathbb{A}^T \dot{s})}{\partial g}. \tag{8}$$

According to (4), and using the mentioned relationships (6) - (8), the equations of motion in terms of the constrained Lagrangian L_c are given by

$$\frac{d}{dt} \left(\partial_{\dot{s}}^T L_c \right) - \partial_s^T L_c + \mathbb{A} \, \partial_g^T L_c = \tau - \mathbb{B} \, \partial_{\dot{g}}^T L, \tag{9}$$

where

$$\begin{aligned} \mathbb{B} \, \partial_{\dot{g}}^T L &= \frac{d}{dt} \left(\mathbb{A} \, \partial_{\dot{g}}^T L \right) - \mathbb{A} \frac{d}{dt} \left(\partial_{\dot{g}}^T L \right) \\ &\quad + \left(\mathbb{A} \, \partial_g^T (\mathbb{A}^T \dot{s}) - \partial_s^T (\mathbb{A}^T \dot{s}) \right) \partial_{\dot{g}}^T L \\ &\Rightarrow \mathbb{B} = \dot{\mathbb{A}} - \partial_s^T (\mathbb{A}^T \dot{s}) + \mathbb{A} \, \partial_g^T (\mathbb{A}^T \dot{s}). \end{aligned} \tag{10}$$

[3] The reader is referred to the references for detailed information regarding Ehresmann connections

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