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# What has Instrumental Variable method to offer for system identification?

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**Abstract:** This paper gathers several experiences of using instrumental variable method in different contexts: closed-loop system identification, LPV model, frequency domain framework.



*Keywords:* instrumental variable, system identification, closed-loop, LPV model, frequency domain

## 1. INTRODUCTION

Mathematical models of dynamic systems are required in most area of scientific enquiry and take various forms, such as differential equations, difference equations, state-space equations and transfer functions. The most widely used approach to mathematical modeling involves the construction of mathematical equations based on physical laws that are known to govern the behaviour of the system. While the advantage of these methods relies on the deep physical insight of the resulting model, their main drawback is the complexity of the model that makes them difficult to be used in applications such as control system design, prediction or decision making.

An alternative to physically-based mathematical modeling is the so-called data-based system identification, which can be applied to any system where experimental data are available. A large scope of system identification approaches has been developed over the past decades. Amongst these, we can cite the prediction error and maximum-likelihood frameworks (see *e.g.* Ljung (1999); Söderström and Stoica (1989); Young (2011)), the subspace-based identification (see *e.g.* Van Overschee and De Moor (1996); Katayama (2005)), the frequency-domain identification (see *e.g.* Pintelon and Schoukens (2001); McKelvey (2002)), the closed-loop identification case (see *e.g.* Van den Hof (1998); Forssell and Ljung (1999); Ninness and Hjalmarsson (2005); Gilson and Van den Hof (2005)).

Most physical systems are continuous-time (CT) whereas, mainly due to the advent of digital computers, research on system identification has concentrated on discrete-time (DT) models from underlying CT systems input/output samples. Recently, interest in identification of CT systems from DT data has arisen (see e.g. Sinha and Rao (1991); Unbehauen and Rao (1987); Garnier and Wang (2008) and references herein) and offer a clever solution in many cases such as irregularly sampled data.

Moreover, systems encountered in practice are often nonlinear or present a time-varying nature. Unlike linearity, non-linearity is a non-property and therefore, non-linearity cannot be defined in a general way. A common framework for the identification of nonlinear models has nevertheless been presented in Sjöberg et al. (1995) and Juditsky et al. (1995). Usually, nonlinear models are classified into two classes: non-parametric models and parametric models. However, another type of models has more recently arose the attention of the system identification community and form an intermediate step between Linear Time-Invariant (LTI) systems and nonlinear/time-varying plants: the model class of Linear Parameter-Varying (LPV) systems (Bamieh and Giarré (2002); Tóth (2010)).

When considering methods that can be used to identify (linear or non linear, CT or DT) models of systems operating in open- or closed-loop, instrumental variable (IV) techniques are rather attractive since they are normally simple or iterative modifications of the linear regression algorithm. For instance, when dealing with complex processes, it can be attractive to rely on methods, such as these, that do not require non-convex optimization algorithms. In addition to this computationally attractive property, IV methods also have the potential advantage that they can yield consistent and asymptotically unbiased estimates of the plant model parameters if the noise does not have rational spectral density or if the noise model is mis-specified; or even if the control system is non-linear and/or time-varying, in the closed-loop framework (Gilson and Van den Hof (2005); Gilson et al. (2011)). Even if several works arise these last ten years (*e.g.* Young (2011); Dankers et al. (2014); Van Herpen et al. (2014); Laurain et al. (2010); Douma (2006)), IV methods have not yet really received the attention that it deserves.

This paper is dedicated to the use of IV methods in several cases of system identification. After an introduction of the IV principles in Section 2, the focus is made on closed-loop system in Section 3, on LPV models with an application on rainfall-flow modeling in Section 4 and on frequency domain framework in Section 5.

## 2. INSTRUMENTAL VARIABLE METHOD

System identification is based on three main ingredients: data (experiment design), model set selection, identification criterion, which are used to estimate a model of a given system. In this paper, we will mainly focus on the identification criterion named Instrumental Variable (IV).

IV is a criterion aiming at minimizing the prediction error. Consider a stable, linear, Single Input Single Output (SISO) data-generating system assumed to be described as

$$\mathcal{S}: \quad y(t_k) = G_0(q)u(t_k) + H_0(q)e(t_k) \tag{1}$$

The plant is denoted by $G_0(q) = B_0(q^{-1})/A_0(q^{-1})$ with the numerator and denominator degree equals to $n_0$, $q^{-1}$ is the


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delay operator with $q^{-i}x(t_k) = x(t_{k-i})$. $u$ describes the plant input signal, $y$ the plant output signal. A colored disturbance $\xi_0(t_k) = H_0(q)e_0(t_k)$ is assumed to affect the system, where $e_0$ is a white noise, with zero mean and variance $\sigma^2_{e_0}$.

The following general model structure and parameter plant model are chosen to model the system

$$\mathcal{M}: \quad y(t_k) = G(q,\theta)u(t_k) + H(q,\theta)e(t_k) \tag{2}$$

$$\mathcal{G}: \quad G(q,\theta) = \frac{B(q^{-1},\theta)}{A(q^{-1},\theta)} \tag{3}$$

In the prediction error method (PEM), the parameters are computed by minimizing the criterion function (see Ljung (1999))

$$V(q,\theta) = \frac{1}{N}\sum_{k=1}^{N}[\varepsilon(t_{k,\theta})]^2 \tag{4}$$

where $\varepsilon(t_{k,\theta}) = y(t_k) - \hat{y}(t_k,\theta)$ is the prediction error. Therefore, the parameters are given as

$$\hat{\theta} = \arg\min_{\theta} \frac{1}{N}\sum_{k=1}^{N}\left(y(t_k) - \hat{y}(t_k,\theta)\right)^2 \tag{5}$$

It has to be noted that the estimation of $\hat{\theta}$ might be a non convex optimization problem for a general nonlinear one-step-ahead predictor. However, the problem (5) can be simplified, for *e.g.*, by choosing an adequate model structure. As a result, for a linear regression, $\hat{\theta}$ is provided by solving the LS solution where

$$\hat{\theta}_{ls} = \arg\min_{\theta} \sum_{k=1}^{N}\left(y(t_k) - \varphi^T(t_k)\theta\right)^2 \tag{6}$$

with $\varphi(t_k)$ denotes the regressor.
The other solution is to use the IV criterion where its basic version aims at computing the estimate $\hat{\theta}$ by solving (Söderström and Stoica (1983))

$$\hat{\theta}_{biv} = \text{sol}\left\{\frac{1}{N}\sum_{k=1}^{N}\zeta(t_k)\left(y(t_k) - \varphi^T(t_k)\theta\right) = 0\right\} \tag{7}$$

where $\zeta(t_k)$ is the so-called instrument. There is a large amount of freedom in the choice of the instrument. It should be correlated with the data but uncorrelated with the noise.This idea has been generalized to the extended IV framework where

$$\hat{\theta}_{xiv} = \arg\min_{\theta}\left\|\left[\frac{1}{N}\sum_{k=1}^{N}L(q)\zeta(t_k)L(q)\varphi^T(t_k)\right]\theta - \left[\frac{1}{N}\sum_{k=1}^{N}L(q)\zeta(t_k)L(q)y(t_k)\right]\right\|_W^2, \tag{8}$$

where $\zeta(t) \in \mathbb{R}^{n_\zeta}$ with $n_\zeta \geq 2n$, $\|x\|^2_W = x^TWx$, with $W$ a positive definite weighting matrix and $L(q)$ a stable prefilter.

By definition, when $G_0 \in \mathcal{G}$, the extended-IV estimate is consistent under the following two conditions [1]

- $\bar{\mathbb{E}}L(q)\zeta(t_k)L(q)\varphi^T(t_k)$ is full column rank,
- $\bar{\mathbb{E}}L(q)\zeta(t_k)L(q)v_0(t_k) = 0$.

The interesting property of the IV methods is that they provide asymptotically unbiased estimates even if the noise is miss-specified. However, the choice of $\zeta(t_k)$, $n_\zeta$, $W$ and the prefilter $L(q)$ may have a considerable effect on the covariance matrix.

[1] The notation $\bar{\mathbb{E}}[.] = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\mathbb{E}[.]$ is adopted from the prediction error framework of Ljung (1999).

The optimal IV algorithm providing the minimum value of the covariance matrix is known to be obtained for (see Söderström and Stoica (1983); Young (2011, 2014))

$$\mathring{\varphi}_f(t_k) = L^{opt}(q)\mathring{\varphi}(t_k), \tag{9}$$

$$L^{opt}(q) = \frac{1}{A_0(q^{-1})H_0(q)}, \text{ and } \zeta(t_k) = \mathring{\varphi}(t_k). \tag{10}$$

where $\mathring{\varphi}(t_k)$ is the noise-free part of $\varphi(t_k)$. Using equations (8) and (9)-(10), the following IV estimate is optimal

$$\hat{\theta}^{opt}_{iv}(N) = \left(\sum_{t=1}^{N}\zeta_f(t_k)\varphi_f^T(t_k)\right)^{-1}\left(\sum_{t=1}^{N}\zeta_f(t)y_f(t)\right) \tag{11}$$

and where the regressor $\varphi_f(t_k) = L^{opt}(q)\varphi(t_k)$, the output $y_f(t_k) = L^{opt}(q)y(t_k)$ and the instrument vector $\zeta_f(t_k) = L^{opt}(q)\zeta(t_k)$ are filtered by $L^{opt}(q)$ (10).
It has to be noted that in this IV estimator, the optimal choice of instruments and prefilter is dependent on unknown system properties which has to be taken care of with an iterative procedure.

## 3. CLOSED-LOOP SYSTEM IDENTIFICATION

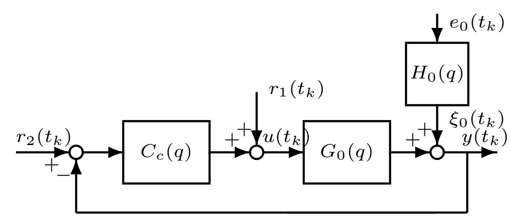


Fig. 1. Closed-loop system configuration

The basic difference between open-loop and closed-loop (CL) system identification is due to the correlation between the input $u(t_k)$ and the noise which conduces the usual open-loop system identification procedure to bias results in the closed-loop context. Therefore several closed-loop methods have been dealt with in the literature and this paper focuses on the IV solution.
Consider a stable, linear, SISO, closed-loop system of the form shown in Figure 1. The data generating system is assumed to be given by the following relations

$$\mathcal{S}: \begin{cases} y(t_k) = G_0(q)u(t_k) + H_0(q)e_0(t_k) \\ u(t_k) = r(t_k) - C_c(q)y(t_k), \end{cases} \tag{12}$$

$$\text{where } r(t_k) = r_1(t_k) + C_c(q)r_2(t_k). \tag{13}$$

The plant is denoted by $G_0(q) = B_0(q^{-1})/A_0(q^{-1})$ with the numerator and denominator degree equals to $n_0$, the controller is denoted by $C_c(q)$. The general model structure and paramatrized plant model are chosen respectively as

$$\mathcal{M}: \; y(t_k) = G(q,\theta)u(t) + H(q,\theta)\varepsilon(t_k,\theta), \tag{14}$$

$$\mathcal{G}: G(q,\rho) = \frac{B(q^{-1},\theta)}{A(q^{-1},\theta)} \tag{15}$$

where $n$ denotes the plant model order and with the pair $(B,A)$ assumed to be coprime.
As for the open-loop situation, the choice of the design variables as the instrument $\zeta(t)$ and the prefilter $L(q)$ have a considerable effect on the covariance matrix produced by the IV estimation algorithm. The covariance properties of the closed-loop IV methods have been investigated in Gilson and Van den Hof (2005) and further insights about the choice of these design