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An improved pseudo-state estimator for a class of commensurate fractional order linear systems based on fractional order modulating functions[☆]

Yan-Qiao Wei [a,b], Da-Yan Liu [a,b,*], Driss Boutat [b], Yi-Ming Chen [a,c,d]

[a] School of Sciences, Yanshan University, Qinhuangdao, Hebei, 066004, PR China
[b] INSA Centre Val de Loire, Université d'Orléans, PRISME EA 4229, Bourges Cedex, 18022, France
[c] LE STUDIUM RESEARCH PROFESSOR, Loire Valley Institute for Advanced Studies, Orléans, France
[d] PRISME (INSA-Institut National des Sciences Appliquées)-88, Boulvevard Lahitolle, 18000 Bourges, France

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ABSTRACT

In this paper, a non-asymptotic pseudo-state estimator for a class of commensurate fractional order linear systems is designed in noisy environment. Different from existing modulating functions methods, the proposed method is based on the system model with fractional sequential derivatives by introducing fractional order modulating functions. By applying the fractional order integration by parts formula and thanks to the properties of the fractional order modulating functions, a set of fractional derivatives and fractional order initial values of the output are analogously obtained by algebraic integral formulas. Then, an explicit formula of the pseudo-state is accomplished by using the fractional sequential derivatives of the output computed based on the previous results. This formula does not contain any source of errors in continuous noise-free case, and can be used to non-asymptotically estimate the pseudo-state in discrete noisy case. The construction of the fractional order modulating functions is also shown, which is independent of the time. Finally, simulations and comparison results demonstrate the efficiency and robustness of the proposed method.

1. Introduction

During recent decades, the studies upon fractional order systems and controllers have been hot topics, and numerous scientific works have been proposed in various aspects [1–5]. Due to the non-locality of fractional operators, the real-state of a fractional order system can be divided into two parts: the pseudo-state and an initialization function [6]. However, the knowledge of the pseudo-state is quite vital in some applications of fractional order systems [6]. Thus, the estimation of the pseudo-state is obviously valuable. Among the previous works, the fractional order observers [7] and the modulating functions method [8] are notable. The fractional order observers are usually asymptotic and cannot meet some requirements of online applications [9,10]. To overcome this drawback, the modulating functions method has been proposed in [8] as a kind of non-asymptotic method, which is also robust against corrupting noises.

Recall that the modulating functions method was first introduced for linear identification in [11] and has been widely used for linear and non-linear identification of integer order systems [12]. Recently, this method has been extended to fractional order case, such as parameter estimation [13,14], pseudo-state estimation [8] and fractional order differentiators [15,16]. In [8], generalized modulating functions were used to derive algebraic formula for the pseudo-state by eliminating undesired terms. Thanks to the algebraic integrals performing as low-pass filters [17], the method was efficiently and robustly executed without knowing the initial conditions. Though, there are still a wide space remaining for further innovation. Actually, in [8] the pseudo-state was estimated by separately estimating the fractional derivatives of the output and some fractional order initial values. For this purpose, a set of fractional order differential equations were constructed, which produced more undesired terms. Thus, some more conditions were imposed to the modulating functions.

Bearing these points in mind, compared to [8], a distinctive modulating functions method is provided in this paper by introducing fractional order modulating functions. Similar to [8], the proposed estimator is also non-asymptotic and robust against

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* Corresponding author at: INSA Centre Val de Loire, Université d'Orléans, PRISME EA 4229, Bourges Cedex, 18022, France.
E-mail addresses: yanqiao.wei@insa-cvl.fr (Y.-Q. Wei), dayan.liu@insa-cvl.fr (D.-Y. Liu), driss.boutat@insa-cvl.fr (D. Boutat), chenym@ysu.edu.cn (Y.-M. Chen).

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noises. Superior to [8], the estimation process is greatly simplified and the computation effort is reduced with the main advantages outlined as follows.

- There is no need to construct additional equations.
- No more additional fractional order initial values are generated.
- The required fractional derivatives and fractional order initial values are analogously estimated.
- The construction of the used modulating functions is independent of time.
- Better robustness with respect to noises than [8] is shown in numerical results.

In the rest parts, the content of this paper is organized as follows. Preliminaries and the problem formulation are provided in Section 2. The advantages and the details of the proposed pseudo-state estimator are presented in Section 3, as well as the construction of the fractional order modulating functions. Numerical example and conclusions are given in the last two sections.

2. Preliminaries

2.1. Fractional calculus

This subsection presents some definitions and properties on fractional calculus, which are useful in this work.

Through this paper, the following notations are used: $I = [0, h] \subset \mathbb{R}$, $\alpha \in \mathbb{R}_+$,[1] and $l = \lceil \alpha \rceil$, where $\lceil \alpha \rceil$ denotes the smallest integer greater than or equal to α. Then, the following definitions can be found in [18,19].

Definition 1. The Riemann–Liouville fractional integral of a function f is defined as follows:

$$\begin{cases} \mathrm{D}_t^0 f(t) & := f(t), \\ \mathrm{D}_t^{-\alpha} f(t) & := \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \end{cases} \tag{1}$$

where $\Gamma(\cdot)$ is the well-known Gamma function [20].

Definition 2. The Riemann–Liouville fractional derivative of a function f is defined as $\mathrm{D}_t^\alpha f(t) := \frac{d^l}{dt^l}\left\{\mathrm{D}_t^{\alpha-l} f(t)\right\}$.

Definition 3. Let $k \in \mathbb{N}$, the Riemann–Liouville fractional sequential derivative of a function f is defined as follows:

$${}_s\mathcal{D}_t^{k\alpha} f(t) := \begin{cases} f(t), & \text{for } k = 0, \\ \mathrm{D}_t^\alpha \left\{ {}_s\mathcal{D}_t^{(k-1)\alpha} f(t) \right\}, & \text{for } k \geq 1. \end{cases} \tag{2}$$

Based on the additive index law of fractional derivatives [18], the following important lemma can be given.

Lemma 1 ([8]). *The Riemann–Liouville fractional integral and derivative of a fractional sequential derivative can be given as follows:* $\forall\, \beta \in \mathbb{R}$, $\forall\, k \in \mathbb{N}^*$,

$$\mathrm{D}_t^\beta \left\{ {}_s\mathcal{D}_t^{k\alpha} f(t) \right\} = \mathrm{D}_t^{\beta+k\alpha} f(t) - \phi_{\beta,k,\alpha}\left\{f(t)\right\}, \tag{3}$$

where

$$\phi_{\beta,k,\alpha}\left\{f(t)\right\} := \sum_{j=1}^{k} \psi_{\beta+(j-1)\alpha,\alpha} \left\{ {}_s\mathcal{D}_t^{(k-j)\alpha} f(t) \right\}. \tag{4}$$

$\psi_{\beta,\alpha}\left\{f(t)\right\}$ *is a decreasing function of* t *defined by:*

$$\psi_{\beta,\alpha}\left\{f(t)\right\} := \sum_{i=1}^{\lceil \alpha \rceil} c_{\beta,i}\, t^{-\beta-i} \left[\mathrm{D}_t^{\alpha-i} f(t)\right]_{t=0} \tag{5}$$

with

$$c_{\beta,i} = \begin{cases} 0, & \text{if } \beta \in \mathbb{Z}, \\ \dfrac{1}{\Gamma(1-\beta-i)}, & \text{else.} \end{cases} \tag{6}$$

Moreover, we have: $\forall\, k \geq 2$,

$${}_s\mathcal{D}_t^{k\alpha} f(t) = \mathrm{D}_t^{k\alpha} f(t) - \phi_{\alpha,(k-1),\alpha}\left\{f(t)\right\}. \tag{7}$$

Consequently, (7) provides the relationship between the derivatives introduced in Definitions 2 and 3. Let us introduce some other kinds of fractional derivatives with useful formulas.

Definition 4 ([19]). The right-sided Caputo fractional derivative of a function f is defined on $[0, h[$ as follows:

$$\begin{cases} {}^C\mathrm{D}_{t,h}^0 f(t): & = f(t), \\ {}^C\mathrm{D}_{t,h}^\alpha f(t): & = \dfrac{(-1)^l}{\Gamma(l-\alpha)} \displaystyle\int_t^h (\tau-t)^{l-\alpha-1} f^{(l)}(\tau)\, d\tau. \end{cases} \tag{8}$$

The fractional order integration by parts formula is an indispensable tool in this work, which is given in the following lemma.

Lemma 2 ([21]). *For any interval* $[0, t] \subset I$, *the following formula holds:*

$$\int_0^t g(\tau) \mathrm{D}_\tau^\alpha f(\tau)\, d\tau = \int_0^t {}^C\mathrm{D}_{\tau,t}^\alpha g(\tau) f(\tau)\, d\tau + \sum_{k=0}^{\lceil \alpha \rceil - 1} (-1)^k \left[g^{(k)}(\tau) \mathrm{D}_\tau^{\alpha-1-k} f(\tau) \right]_{\tau=0}^{\tau=t}. \tag{9}$$

Finally, the following fractional derivative will be quite important in the sequel.

Definition 5. The right-sided Caputo fractional sequential derivative of a function f is defined as follows:

$${}_s^C\mathcal{D}_{t,h}^{k\alpha} f(t) := \begin{cases} f(t), & \text{for } k = 0, \\ {}^C\mathrm{D}_{t,h}^\alpha \left\{ {}_s^C\mathcal{D}_{t,h}^{(k-1)\alpha} f(t) \right\}, & \text{for } k \geq 1. \end{cases} \tag{10}$$

Remark that several kinds of fractional derivatives are previously defined for different purposes. First, the considered fractional order systems are defined based on the Riemann–Liouville fractional derivatives. Second, the pseudo-state will be expressed by the fractional sequential derivatives of the output. Thanks to Lemma 1, the latter will be obtained by calculating the Riemann–Liouville fractional derivatives and some fractional order initial values of the output. Finally, instead of calculating directly the Riemann–Liouville fractional derivatives of the output, algebraic integral formulas will be provided by applying Lemma 2, where the right-sided Caputo fractional derivatives of constructed modulating functions need to be explicitly calculated.

2.2. Problem formulation

Let us consider the following commensurate fractional order linear system within this framework:

$$\mathrm{D}_t^\alpha x = Ax + Bu, \tag{11}$$

$$y = Cx, \tag{12}$$

on $I \subset \mathbb{R}_+ \cup \{0\}$, where $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times 1}$, $C \in \mathbb{R}^{N\times 1}$, $\mathrm{D}_t^\alpha x = \left(\mathrm{D}_t^\alpha x_1, \ldots, \mathrm{D}_t^\alpha x_N\right)^T$ with $\alpha = \frac{1}{q}$, $q, N \in \mathbb{N}^*$, $x \in \mathbb{R}^N$ is the

[1] In this paper, $\mathbb{R}_+$ denotes the set of positive real numbers, and $\mathbb{N}^*$ denotes the set of positive integers.