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A quiver approach to studying orbit spaces of linear systems

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ABSTRACT

Orbit spaces associated to linear actions are of particular interest in control theory. Their geometrical properties can be naturally investigated by using the representations of quivers as an abstract framework. The aim of the paper is to bring into attention an application of this approach and to show how the use of quivers makes it easy handling concepts arising in control theory. Specifically, the natural duality between controllable and observable systems, as well as the construction of compactifications for the associated orbit spaces is interpreted in terms of opposite quivers.

1. Introduction

The use of techniques from algebraic geometry has been proved to be a very fruitful approach in solving problems arising in control theory. In the late '70s, several pioneering works (e.g. those by Hazewinkel and Kalman [1] or by Byrnes and Hurt [2]) highlighted some meaningful algebro-geometric tools in constructing moduli spaces for linear dynamical systems; an excellent synthesis of these techniques, as well as a source for relevant references is the monograph by Tannenbaum [3]. Recent developments (see [4,5]) brought into attention the opportunity of using (partial) representations of quivers as framework for constructing moduli spaces for certain classes of linear systems or for studying their topological properties. Moreover, it was proved that the compactification proposed by Helmke and Shayman in [6] for the orbit space of controllable state space systems and that one proposed by Helmke (see [7,8]) for the space of rational transfer functions can be interpreted in terms of orbit spaces associated to quiver representations. Actually, these orbit spaces are constructed by adapting ideas from Mumford's Geometric Invariant Theory (see [9]) to the case of linear actions, particularly to the case of representations of quivers (see [10,11]). Furthermore, it was shown in [12] that, by appropriately adapting the concepts and the definitions, all these results can be obtained by completely remaining within the framework of linear algebra.

The aim of this note is to bring into attention other applications of the quiver approach for problems arising in control theory. The paper is organized as follows. Section 2 is dedicated to revising the main concepts referring to stability, respectively to the construction of orbit spaces associated to quiver factorization problems. Section 3 is divided in two parts. Section 3.1 deals with results that are abstract in nature, referring to the natural relationship between stability concepts (Proposition 1), respectively between orbit spaces (Theorem 1) associated to opposite quivers. Section 3.2 is oriented towards applications. The well-known duality between controllability and observability is naturally interpreted in Proposition 2 in terms of quiver representations. The interpretation is extended in Theorem 2 for the compactifications of the associated orbit spaces of controllable and observable linear systems.

2. Orbit spaces associated to quiver representations

Quiver factorization problems. Let $Q = (N, A, h, t)$ be a quiver (here N and A are the sets of nodes, respectively of arrows, while $h, t : A \to N$ are the head, respectively the tail maps). A representation $r = (\mathbf{V}, \psi)$ of Q over the field $\mathbb{K}$ ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) is a pair consisting of a family of $\mathbb{K}$-vector spaces $\mathbf{V} = (V_\nu)_{\nu\in N}$ (assumed throughout the paper to be finite dimensional) and a family of linear maps $(\psi_a)_{a\in A}$. The set of all representations of Q having the same family of vector spaces $\mathbf{V}$ as support is

$$W_{\mathbf{V}} = \bigoplus_{a\in A} \mathrm{Hom}(V_{t(a)}, V_{h(a)}).$$

The group $\prod_{\nu\in N} \mathrm{GL}(V_\nu)$ acts naturally on the space $W_{\mathbf{V}}$, by

$$(g_\nu)_\nu \cdot (\psi_a)_a := (g_{h(a)} \circ \psi_a \circ g_{t(a)}^{-1})_a.$$

However, in some situations considering a smaller symmetry group may provide geometric meaningful constructions (see e.g.

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Example 2 below or the examples in Section 3.2). Let therefore M be a non-empty subset of nodes $\emptyset \neq M \subseteq N$ and denote by G_M the product $\prod_{\nu\in M} \mathrm{GL}(V_\nu)$. According to [4], the pair (Q, M) is a *quiver with marked vertices* and, using the terminology from [13], the problem of constructing orbit spaces for the G_M-action on $W_{\mathbf{V}}$ is called a *quiver factorization problem*.

Elements of Hermitian type. An element $\mathbf{s}$ of the Lie algebra $\mathfrak{g}_M$ of G_M is of Hermitian type if there exist a compact subgroup K_M of G_M and a Cartan-type decomposition $\mathfrak{g}_M = \mathfrak{k}_M \oplus \mathfrak{p}_M$ (see e.g. [14] for the terminology) such that $\mathbf{s} \in \mathfrak{p}_M$. The case $\mathbb{K} = \mathbb{C}$ was discussed in [11, Definition 3.1] and, in this case, the equality $\mathfrak{p}_M = i\mathfrak{k}_M$ holds—as in the decomposition $\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n) \oplus i\mathfrak{u}(n)$ of the vector space $\mathfrak{gl}(n, \mathbb{C})$ as a direct sum between the vector space of unitary matrices and that one of Hermitian matrices. In the case $\mathbb{K} = \mathbb{R}$, there is no longer such a relationship between $\mathfrak{k}_M$ and $\mathfrak{p}_M$—as in the decomposition $\mathfrak{gl}(n, \mathbb{R}) = \mathfrak{o}(n) \oplus \mathrm{sym}(n)$ of the vector space $\mathfrak{gl}(n, \mathbb{R})$ as a direct sum between the vector space of orthogonal matrices and that one of symmetric matrices. The main point is that, in both cases, if $\mathbf{s}$ is of Hermitian type, then $\mathbf{s}$ has only real eigenvalues and it is diagonalizable. Furthermore, according to [11, Definition 3.1], if $\rho : G_M \to \mathrm{GL}(W)$ is a representation of G_M, then $\rho_*(\mathbf{s})$ has only real eigenvalues (here $\rho_* : \mathfrak{g}_M \to \mathfrak{gl}(W)$ is the derivative of ρ). We will denote by $H(G_M)$ the set of elements of Hermitian type corresponding to the group G_M. The following remark is a key ingredient for handling them. It refers only to the case when the group G_M has the form $\mathrm{GL}(V)$, but it naturally extends to the general case.

Remark 1. Let V be a $\mathbb{K}$-vector space ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) and let $s \in \mathfrak{gl}(V)$ be an element of Hermitian type. Then s gives rise to the following data.

- A sequence of real numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_q$ (the eigenvalues of s), together with the corresponding multiplicities $m_{\lambda_1}, \ldots, m_{\lambda_q}$.
- A filtration $\mathcal{F}_s : \{0\} = V_0 \subset V_1 \subset \cdots \subset V_q = V$ of V, provided by the eigenspaces of s. We denoted $V_i = \oplus_{\lambda \le \lambda_i} V(\lambda)$, where $V(\lambda)$ is the eigenspace corresponding to the eigenvalue λ. It obviously holds the relation $\dim_{\mathbb{K}} V(\lambda) = m_\lambda$.

It is worth to notice that $\mathrm{tr}(s) = \sum_{i=1}^{q} m_{\lambda_i}\lambda_i$, that is the trace of s depends only on the eigenvalues of s and on the dimensions of the vector spaces arising in the filtration $\mathcal{F}_s$. In particular, let $g \in \mathrm{GL}(V)$. Define a new element of Hermitian type $g * s$ as follows. Chose a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$, where $n = \dim_{\mathbb{K}} V$, such that s has diagonal form with respect to $\mathcal{B}$ and define $g * s$ such that it has the same diagonal form with respect to the basis $\{g \cdot b_1, \ldots, g \cdot b_n\}$. The eigenvalues of $g * s$ and their multiplicities are equal to those of s, while the filtration of $g * s$ is given by the inclusions $\{0\} \subset g(V_1) \subset g(V_2) \subset \cdots \subset g(V_q)$. Although the filtrations associated to s and to $(g * s)$ do not coincide, the dimensions of the corresponding subspaces are equal and, particularly, one has $\mathrm{tr}\, s = \mathrm{tr}\,(g * s)$.

Weights. For defining the concept of weight we first consider the case when $G_M = \mathrm{GL}(V)$, that is M contains only one node. Let $\mathcal{T}_{G_M}$ be the one-dimensional $\mathbb{K}$-vector subspace of $\mathfrak{g}_M^\vee$ generated by the trace, that is by the functional $\xi \mapsto \mathrm{tr}(\xi)$ ($\mathfrak{g}_M^\vee$ denotes the dual of the Lie algebra $\mathfrak{g}_M$). For any Cartan-type decomposition $\mathfrak{g}_M = \mathfrak{k}_M \oplus \mathfrak{p}_M$, by restricting the elements of $\mathcal{T}_{G_M}$ to $\mathfrak{p}_M$, one gets real multiples of trace (independently if $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and independently on the chosen Cartan-type decomposition). These functionals will be called in the sequel *weights*. We notice that the convention used here is slightly different from that used in [11,12]: in these papers, where only the complex case is treated, weights do actually belong to $\mathfrak{k}_M^\vee$ and the identification $\mathfrak{p}_M = i\mathfrak{k}_M$ makes it possible to relate weights and elements of Hermitian type. The definition proposed above has the advantage to be applicable both in real and complex cases and, via natural identifications, it can be related to the previous one. Summing up, in the sequel a weight is nothing else but a functional θ of the form $\theta = \alpha\mathrm{tr}(\cdot), \alpha \in \mathbb{R}$, acting on elements of Hermitian type. For an arbitrary set of nodes M, giving a weight $\theta \in \mathcal{T}_{G_M}$ for the group G_M is equivalent to giving a family of weights $(\theta_\nu)_{\nu\in M}$.

Stability. Throughout this paper we will use the concept of θ-(semi)stability, where $\theta \in \mathcal{T}_{G_M}$. This definition extends that one from [11], where only the case $\mathbb{K} = \mathbb{C}$ is considered. An element $w \in W_{\mathbf{V}}$ is called *θ-semistable* if for any $\mathbf{s} \in H(G_M)$ such that $w \in W_{\mathbf{V}}^{\le 0}(\mathbf{s})$, it holds $\langle\theta, \mathbf{s}\rangle \ge 0$. If w is θ-semistable and, furthermore, for any $\mathbf{s} \in H(G_M) \setminus \mathfrak{h}$ such that $w \in W_{\mathbf{V}}^{\le 0}(\mathbf{s})$ it holds $\langle\theta, \mathbf{s}\rangle > 0$, the element w is called *θ-stable*. Here $\mathfrak{h}$ is the Lie algebra of the kernel of the representation, while $W_{\mathbf{V}}^{\le 0}(\mathbf{s})$ is the vector space spanned by eigenvectors corresponding to nonpositive eigenvalues of $\rho_*(\mathbf{s})$ (notice that, since $\mathbf{s}$ is an element of Hermitian type, as noticed in Remark 1, the eigenvalues of $\rho_*(\mathbf{s})$ are all real). Let us first notice that stability is invariant under multiplication by a positive constant, i.e. if w is θ-stable, then it is $\alpha\theta$-stable for any positive constant α. We deduce that for each node there exist essentially three different stability conditions, corresponding to the weights $\theta_0 := 0, \theta_1 := \mathrm{tr}, \theta_{-1} := -\mathrm{tr}$. In conclusion, giving a weight $\theta \in \mathcal{T}_{G_M}$ for the group G_M is equivalent to giving a family of weights $(\theta_{\varepsilon_\nu})_{\nu\in M}$, where for each marked node $\nu \in M$ one has $\varepsilon_\nu \in \{-1, 0, 1\}$. Furthermore, for any element of Hermitian type $\mathbf{s} = (s_\nu)_{\nu\in M}$, one has $\langle\theta, \mathbf{s}\rangle = \sum_{\nu\in M}\langle\theta_{\varepsilon_\nu}, s_\nu\rangle$. The sets of all θ-(semi)stable points will be denoted by $W_{\mathbf{V}}^{(s)s,\theta}$. We claim that these sets are unions of orbits. Indeed, let us fix an element $g \in G_M$ of the symmetry group G_M and a vector $w \in W_{\mathbf{V}}$. By Remark 1, one has that $w \in W_{\mathbf{V}}^{\le 0}(\mathbf{s})$ if and only if $g \cdot w \in W_{\mathbf{V}}^{\le 0}(g * \mathbf{s})$. On the other hand, again by Remark 1, we deduce that $\langle\theta, \mathbf{s}\rangle = \langle\theta, g * \mathbf{s}\rangle$. In conclusion, w is θ-(semi)stable if and only if $g \cdot w$ is θ-(semi)stable and this yields the desired conclusion.

About the meaning of 'stability'. The approach used throughout this paper for dealing with (semi)stability is self-contained, uses only linear algebra concepts and addresses both the real and the complex case. It is however, worth to notice that its roots go back to Mumford's GIT (see [9]). In that framework, as pointed out in [15, Definition 1.25], stability refers to a specific behavior under a group action: stable points have closed orbits and finite stabilizer. In the complex case, the definition proposed above is equivalent to the original GIT definition, as applied in [10] to linear actions. The result relies on the Kempf–Ness theorem in [16] relating stability to the minima of a certain norm; for a comprehensive presentation and for further references, we refer the interested reader to [11]. In the real case, recent results (e.g. [17,18]) indicate that the Kempf–Ness theorem still holds and one expects to have an analogous interpretation of the definition given above. Anyway, the proofs presented in the sequel are independent on these relationships.

QFP-quotients. According to the general theory (see e.g. [10,11]), the restriction of the action on the space of θ-semistable orbits provides a quotient with 'nice' geometric properties for the G_M-action. Furthermore, it is easy to see that, up to homeomorphism, such a quotient (called *QFP-quotient*) depends only on the combinatorial data (Q, M), the dimension vector $\mathbf{d} = (d_\nu)_{\nu\in N}$ (where $d_\nu = \dim_{\mathbb{K}} V_\nu$) of the quiver representation and the weight θ. Some topological properties of a QFP-quotient can be easily described using this data. For instance, a non-empty QFP-quotient is compact if and only if Q contains no marked cycles and no oriented paths with unmarked source and sink (see [4]). We will denote by $\mathcal{M}(Q, M, \mathbf{d}, \theta)$ the quotient obtained for the 'standard' family of vector spaces associated to Q and having dimension vector $\mathbf{d}$, i.e. for any node ν one puts $V_\nu = \mathbb{K}^{d_\nu}$.

We finish this preparatory subsection by describing two examples. The first one aims to detail the use of the quiver approach in