Systems & Control Letters 73 (2014) 42–47

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

On the eigenvalue decay of solutions to operator Lyapunov equations

Luka Grubišić [a,*], Daniel Kressner [b]

[a] University of Zagreb, Department of Mathematics, Bijenička 30, 10000 Zagreb, Croatia

[b] EPF Lausanne, SB-MATHICSE-ANCHP, Station 8, CH-1015 Lausanne, Switzerland

ARTICLE INFO

Article history:
Received 17 February 2014
Received in revised form
1 September 2014
Accepted 12 September 2014

Keywords:
Balanced truncation
Exponential decay
Lyapunov equation

ABSTRACT

This paper is concerned with the eigenvalue decay of the solution to operator Lyapunov equations with right-hand sides of finite rank. We show that the kth (generalized) eigenvalue decays exponentially in $\sqrt{k}$, provided that the involved operator A generates an exponentially stable analytic semigroup, and A is either self-adjoint or diagonalizable with its eigenvalues contained in a strip around the real axis. Numerical experiments with discretizations of 1D and 2D PDE control problems confirm this decay.

1. Introduction

The Lyapunov matrix equation

$$AX + XA^T = -BB^T \tag{1.1}$$

with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ plays a central role in balanced truncation model reduction for linear time-invariant control systems [1]. Assuming that A is stable (i.e., all its eigenvalues have negative real part), Eq. (1.1) has a unique, bounded, nonnegative, and self-adjoint solution X. Typically, the eigenvalues of X decay very quickly when the right-hand side has low rank, that is, $m \ll n$. This decay property is strongly linked to the approximation error attained by balanced truncation as well as the performance of low-rank methods for solving (1.1). Consequently, a number of works [2–7] have studied this decay and derived a priori estimates.

By now, the situation is fairly well understood for a symmetric negative definite matrix A. In this case, it can be shown [7,5] that there is a matrix X_k of rank km such that

$$\|X - X_k\|_F \le \frac{8\|B\|_F}{|\lambda_{\max}(A)|} \exp\left(\frac{-k\pi^2}{\log(8\,\kappa(A))}\right), \tag{1.2}$$

where $\lambda_{\max}(A)$ denotes the largest eigenvalue and $\kappa(A)$ the condition number of A. By the Eckart–Young theorem, this estimate implies that the sorted eigenvalues $\lambda_1(X) \ge \lambda_2(X) \ge \cdots \ge \lambda_n(X)$ of X decay exponentially:

$$\lambda_k(X) \lesssim \gamma^k \quad \text{with } \gamma = \exp\left(\frac{-\pi^2}{m\log(8\,\kappa(A))}\right). \tag{1.3}$$

This bound bears the disadvantage that it deteriorates as $\kappa(A) \to \infty$, a situation of practical relevance when A comes from the (increasingly refined) discretization of an unbounded operator. Indeed, the numerical calculations for an example in Section 5 seem to indicate that the *exponential* decay property gets lost as $\kappa(A) \to \infty$. In fact, the decay is observed to be exponential with respect to $\sqrt{k}$, instead of k. We analyze the generalized eigenvalues of X in the scale of Hilbert spaces associated to A. The aim of this paper is to prove this property for the underlying operator Lyapunov equation, when A has eigenvalues contained in a strip around the real axis and is diagonalizable, and B has finite rank. Our result extends related work by Opmeer [8], which implies superpolynomial decay.

2. Preliminaries

In this section, we will formalize the notation and point out some of the conventions that will be used in this paper.

Given a Gelfand triple $\mathcal{X} \subset \mathcal{H} \subset \mathcal{Z}$ of Hilbert spaces, where $\mathcal{Z} = \mathcal{X}'$ is the dual space to $\mathcal{X}$, we consider a bounded operator A from $\mathcal{X}$ to $\mathcal{Z}$ with a bounded inverse. We let $A' : \mathcal{Z}' \to \mathcal{X}'$ denote the dual operator to A in the duality paring $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{\mathcal{Z}\times\mathcal{X}}$. After identifying biduals we may also write $A' : \mathcal{X} \to \mathcal{Z}$. We will use the notation $\mathcal{X} = \mathrm{Dom}(A)$, since A can also be interpreted as an unbounded operator on $\mathcal{H}$, and then we will denote its domain of definition as $\mathrm{Dom}_{\mathcal{H}}(A) = \{u \in \mathcal{H} : \|Au\|_{\mathcal{H}} < \infty\}$. Moreover, we

* Corresponding author. Tel.: +385 14605881.
E-mail addresses: luka.grubisic@math.hr (L. Grubišić), daniel.kressner@epfl.ch (D. Kressner).

http://dx.doi.org/10.1016/j.sysconle.2014.09.006
0167-6911/

consider a (not necessarily bounded) linear operator $B : \mathcal{U} \to \mathcal{Z}$ for a Hilbert space $\mathcal{U}$ with inner product $(\cdot,\cdot)_{\mathcal{U}}$.

The operators A, B give rise to the *Lyapunov operator equation* in a linear operator X:

$$AX + XA' = -BB', \tag{2.1}$$

which formally stands for the variational formulation

$$\langle Xz_1, A'z_2\rangle_{\mathcal{Z}\times\mathcal{X}} + \langle A'z_1, Xz_2\rangle_{\mathcal{Z}\times\mathcal{X}} = \mathfrak{b}(z_1, z_2), \quad z_1, z_2 \in \mathcal{X} \tag{2.2}$$

with the sesquilinear form $\mathfrak{b}(z_1,z_2) := -(B'z_1, B'z_2)_{\mathcal{U}}$. We refer to, e.g., [9–12] for a more detailed discussion of this equation.

Example 2.1 (*[13]*)**.** Consider the point-wise control of a diffusion process on the interval [0, 1]:

$$z_t(t,x) = \kappa z_{xx}(t,x) + \delta(x - x_b)u(t), \qquad z(x,0) \equiv 0, \tag{2.3}$$

$$y(t) = z(t,x_c), \qquad z(0,t) = z(1,t) = 0, \tag{2.4}$$

where $\kappa > 0$ is the diffusion coefficient and $0 < x_b < x_c < 1$. To set up the operator Lyapunov equation (2.1) for the controllability Gramian, we choose the usual Sobolev spaces $\mathcal{X} = H_0^1(0,1)$, $\mathcal{H} = L^2(0,1)$, and $\mathcal{Z} = H^{-1}(0,1)$. Then $A = \partial_{xx}$ and B is defined by $B : u \mapsto u\,\delta(x - x_b)$ for $u \in \mathbb{R}$. ⋄

Let us assume that A is the infinitesimal generator of an exponentially stable analytic semigroup $(\exp(tA))_{t\ge 0}$ on $\mathcal{H}$. The results of [12, Chapter 5] imply the existence and uniqueness of a bounded nonnegative self-adjoint solution $X : \mathcal{H} \to \mathcal{H}$ to the Lyapunov equation (2.1), provided that A^{-1} is compact and $A^{-1}B$ is bounded. Furthermore, under the additional assumption that $A^{-1}B$ has finite rank Opmeer [8] has proved that X is not only bounded but also contained in every Schatten class [14].

2.1. Choice of Hilbert spaces

Instead of general Hilbert spaces $\mathcal{X}$ and $\mathcal{Z}$, we will use interpolation spaces associated with A. For this purpose, we work with the restricted operator $A : \mathrm{Dom}_{\mathcal{H}}(A) \subset \mathcal{X} \to \mathcal{H}$, which admits the adjoint A^*. Additionally we will assume that A possesses a Riesz basis of eigenvectors $\{\psi_i\}_{i\in\mathbb{N}}$ in $\mathcal{H}$ with associated eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$, this implies $\sup_{i\in\mathbb{N}} \operatorname{Re}\lambda_i < 0$. The Riesz property allows us to represent every $f \in \mathcal{H}$ as

$$f = \sum_{i\in\mathbb{N}} (f,\phi_i)\,\psi_i = \sum_{i\in\mathbb{N}} (f,\psi_i)\,\phi_i,$$

where $(\cdot,\cdot)$ denotes the scalar product in $\mathcal{H}$ and $\{\phi_i\}_{i\in\mathbb{N}}$ is a sequence of eigenvectors for A^*, normalized such that $(\phi_i,\psi_i) = 1$ for $i \in \mathbb{N}$.

Following [15] we define for every $\alpha \in \mathbb{R}$, the Hilbert space

$$\mathcal{H}_\alpha = \left\{ \sum_{i\in\mathbb{N}} f_i\psi_i \,:\, \{f_i|\lambda_i|^\alpha\}_{i\in\mathbb{N}} \in \ell^2 \right\}$$

with the scalar product

$$(f,g)_\alpha = \sum_{i\in\mathbb{N}} (f,\psi_i)(\psi_i,g)|\lambda_i|^{2\alpha}.$$

It holds $\mathcal{H}_{\alpha_1} \subset \mathcal{H} \subset \mathcal{H}_{\alpha_2}$ whenever $\alpha_2 \le 0 \le \alpha_1$. Analogously, we define the Hilbert space

$$\mathcal{H}_\alpha^d = \left\{ \sum_{i\in\mathbb{N}} f_i\phi_i : \{f_i|\lambda_i|^\alpha\}_{i\in\mathbb{N}} \in l^2(\mathbb{N}) \right\}$$

associated with the adjoint operator A^*.

Using this notation, $\mathcal{H} = \mathcal{H}_0 = \mathcal{H}_0^d$, $\mathrm{Dom}_{\mathcal{H}}(A) = \mathcal{H}_1$, $\mathrm{Dom}(A^*) = \mathcal{H}_1^d$, and we may set $\mathcal{Z} = \mathcal{H}_{-1/2}$. For Example 2.1, where A is the Dirichlet Laplace operator, we simply have $\mathcal{X} = H_0^1(0,1) = \mathcal{H}_{1/2}$ and $\mathcal{Z} = H^{-1}(0,1) = \mathcal{H}_{-1/2}$. The following example covers a more complicated situation.

Example 2.2. Let $A = \partial_x(a\partial_x) + b\partial_x + c$ for (possibly complex valued) functions $a, b, c \in L^\infty(0,1)$. Then Kato's square root theorem [16] yields $H_0^1(0,1) = \mathrm{Dom}_{\mathcal{H}}((-A)^{1/2})$. In the case that a is a real valued function such that there exists a Lipschitz function β with $\partial_x\beta = \frac{b}{2a}$ then A has real eigenvalues and is diagonalizable by the multiplication operator $Q : \psi \mapsto e^\beta\psi$, see [17], and so $H_0^1(0,1) = \mathcal{H}_{1/2} = \mathcal{H}_{1/2}^d$. ⋄

3. Selfadjoint case

We first consider the situation when A is self-adjoint on $\mathcal{H}$, has a compact resolvent and negative eigenvalues. We choose $\mathcal{Z} = \mathcal{H}_{-1/2}$, which is equipped with the scalar product $(\cdot, |A|^{-1}\cdot) = (|A|^{-1/2}\cdot, |A|^{-1/2}\cdot)$, and $\mathcal{X} = \mathcal{H}_{1/2} = \mathrm{Dom}_{\mathcal{H}}(|A|^{1/2})$.

Additionally, we assume that the product $|A|^{-1/2}B$ is bounded. This is equivalent to the assumption that

$$b(\psi,\phi) := -\mathfrak{b}(|A|^{-1/2}\psi, |A|^{-1/2}\phi)$$

is everywhere defined and bounded on $\mathcal{H}$. As discussed in [18], the substitutions $\psi = |A|^{1/2}z_1$ and $\phi = |A|^{1/2}z_2$ then allow us to turn (2.2) into the equivalent equation

$$\big(|A|^{1/2}\psi, X|A|^{-1/2}\phi\big) + (X|A|^{-1/2}\psi, |A|^{1/2}\phi) = b(\psi,\phi), \quad \psi,\phi \in \mathcal{X}. \tag{3.1}$$

3.1. Solution formulas

By [18], Eq. (3.1) has a unique solution $X : \mathcal{X} \to \mathcal{X}$, which admits the representation

$$(\psi, X\phi) = \int_0^\infty b\left(\exp(At)|A|^{1/2}\psi, \exp(At)|A|^{1/2}\phi\right)\,\mathrm{d}t, \quad \psi,\phi \in \mathcal{X}. \tag{3.2}$$

The operator $X : \mathcal{X} \to \mathcal{X}$ can be uniquely extended to a bounded operator $X : \mathcal{H} \to \mathcal{H}$, since the solution of the operator Lyapunov equation is unique and $\mathcal{X}$ is assumed to be dense in $\mathcal{H}$. However, the formula (3.2) only holds for $\psi, \phi \in \mathcal{X}$.

Since A is assumed to have a compact resolvent, there are orthonormal eigenvectors $\{\psi_i\}_{i\in\mathbb{N}}$ associated with the eigenvalues $\lambda_i < 0$ of A that span the whole space $\mathcal{H}$. It follows that the solution $X : \mathcal{H} \to \mathcal{H}$ of (3.1) is equivalently defined by the relation

$$(\psi_i, X\psi_j) = -b(\psi_i,\psi_j)\frac{\sqrt{\lambda_i\lambda_j}}{\lambda_i + \lambda_j}. \tag{3.3}$$

3.2. Low-rank approximation

Motivated by techniques for the finite-dimensional case [4,3,5], we derive low-rank approximations for X from (3.3) via approximating the scalar function $1/z$ by a sum of exponentials. For $z \in \mathbb{C}$ with $\operatorname{Re}(z) < 0$ such an approximation is obtained from numerical quadrature applied to the integral representation $-1/z = \int_0^\infty e^{tz}\,\mathrm{d}t$. Sinc quadrature [19] yields the following approximation, see [3, Lemma 5] and [5, Sec. 5.2].

Lemma 3.1. *Let $k \in \mathbb{N}$ and consider $z \in \mathbb{C}$ with $\operatorname{Re}(z) \le -1$. Defining the quadrature nodes and weights*

$$t_p = \log\Big(\exp(ph_{\mathrm{St}}) + \sqrt{1 + \exp(2ph_{\mathrm{St}})}\Big),$$

$$\omega_p = h_{\mathrm{St}}/\sqrt{1 + \exp(2ph_{\mathrm{St}})}, \quad -k \le p \le k,$$

with $h_{\mathrm{St}} = \pi/\sqrt{k}$, yields the approximation error

$$\left| \int_0^\infty \exp(tz)\,\mathrm{d}t - \sum_{p=-k}^{k} \omega_p \exp(t_p z) \right| \le C_{\mathrm{St}} \exp(|\operatorname{Im}(z)|/\pi)\exp\left(-\pi\sqrt{k}\right). \tag{3.4}$$

The constant C_{St} is independent of z and k.