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Stabilization of hybrid stochastic differential equations by feedback control based on discrete-time state observations[☆]

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ABSTRACT

Recently, Mao (2013) discusses the mean-square exponential stabilization of continuous-time hybrid stochastic differential equations by *feedback controls based on discrete-time state observations*. Mao (2013) also obtains an upper bound on the duration τ between two consecutive state observations. However, it is due to the general technique used there that the bound on τ is not very sharp. In this paper, we will consider a couple of important classes of hybrid SDEs. Making full use of their special features, we will be able to establish a better bound on τ. Our new theory enables us to observe the system state less frequently (so costs less) but still to be able to design the feedback control based on the discrete-time state observations to stabilize the given hybrid SDEs in the sense of mean-square exponential stability.

1. Introduction

Hybrid stochastic differential equations (SDEs) (also known as SDEs with Markovian switching) have been used to model many practical systems where they may experience abrupt changes in their structure and parameters. One of the important issues in the study of hybrid systems is the automatic control, with consequent emphasis being placed on the asymptotic analysis of stability [1–19]. In particular, [20,21] are two of most cited papers (Google citations 447 and 269, respectively) while [22] is the first book in this area (Google citation 496).

Recently, Mao [23] investigates the following stabilization problem by a feedback control based on the discrete-time state observations: consider an unstable hybrid SDE

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dw(t), \tag{1}$$

where $x(t) \in R^n$ is the state, $w(t) = (w_1(t), \dots, w_m(t))^T$ is an m-dimensional Brownian motion, $r(t)$ is a Markov chain (please see Section 2 for the formal definitions) which represents the system mode, and the SDE is in the Itô sense. The aim is to design a feedback control $u(x([t/\tau]\tau), r(t), t)$ in the drift part so that the controlled system

$$dx(t) = \big(f(x(t), r(t), t) + u(x([t/\tau]\tau), r(t), t)\big)dt + g(x(t), r(t), t)dw(t) \tag{2}$$

becomes stable, where $\tau > 0$ is a constant and $[t/\tau]$ is the integer part of t/τ. The key feature here is that the feedback control $u(x([t/\tau]\tau), r(t), t)$ is designed based on the discrete-time observations of the state $x(t)$ at times $0, \tau, 2\tau, \dots$. This is significantly different from the stabilization by a continuous-time (regular) feedback control $u(x(t), r(t), t)$, based on the current state, where the aim is to design $u(x(t), r(t), t)$ in order for the controlled system

$$dx(t) = \big(f(x(t), r(t), t) + u(x(t), r(t), t)\big)dt + g(x(t), r(t), t)dw(t) \tag{3}$$

to be stable. In fact, the regular feedback control requires the continuous observations of the state $x(t)$ for all $t \geq 0$, while the feedback control $u(x([t/\tau]\tau), r(t), t)$ needs only the discrete-time observations of the state $x(t)$ at times $0, \tau, 2\tau, \dots$. The latter is

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clearly more realistic and costs less in practice. To the best knowledge of the authors, Mao [23] is the first paper that studies this stabilization problem by feedback controls based on the discrete-time state observations in the area of SDEs, although the corresponding problem for the deterministic differential equations has been studied by many authors (see e.g. [24–28]).

Mao [23] shows that if continuous-time controlled SDE (3) is mean-square exponentially stable, then so is the discrete-time-state feedback controlled system (2) provided that τ is sufficiently small. This is of course a very general result. However, it is due to the general technique used there that the bound on τ is not very sharp. In this paper, we will consider a couple of important classes of hybrid SDEs. Making full use of their special features, we will be able to establish a better bound on τ.

Mathematically speaking, the key technique in Mao [23] is to compare the discrete-time-state feedback controlled system (2) with the continuous-time controlled SDE (3) and then prove the stability of system (2) by making use of the stability of SDE (3). However, in this paper, we will work directly on the discrete-time-state feedback controlled system (2) itself. To cope with the mixture of the continuous-time state $x(t)$ and the discrete-time state $x([t/\tau]\tau)$ in the system, we have developed some new techniques. Let us begin to develop these new techniques and to establish our new theory.

2. Notation and stabilization problem

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $w(t) = (w_1(t), \dots, w_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. If A is a vector or matrix, its transpose is denoted by A^T. If $x \in R^n$, then $|x|$ is its Euclidean norm. If A is a matrix, we let $|A| = \sqrt{\text{trace}(A^T A)}$ be its trace norm and $\|A\| = \max\{|Ax| : |x| = 1\}$ be the operator norm. If A is a symmetric matrix ($A = A^T$), denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ its smallest and largest eigenvalues, respectively. By $A \le 0$ and $A < 0$, we mean A is non-positive and negative definite, respectively. Denote by $L^2_{\mathcal{F}_t}(R^n)$ the family of all $\mathcal{F}_t$-measurable R^n-valued random variables ξ such that $\mathbb{E}|\xi|^2 < \infty$, where $\mathbb{E}$ is the expectation with respect to the probability measure $\mathbb{P}$. If both a, b are real numbers, then $a \vee b = \min\{a, b\}$ and $a \wedge b = \max\{a, b\}$. Let $r(t)$, $t \ge 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \dots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \ne j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \ge 0$ is the transition rate from i to j if $i \ne j$ while

$$\gamma_{ii} = -\sum_{j\ne i} \gamma_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is known that almost all sample paths of $r(t)$ are constant except for a finite number of simple jumps in any finite subinterval of R_+ $(:= [0, \infty))$. We stress that almost all sample paths of $r(t)$ are right continuous.

Consider an n-dimensional linear hybrid SDE

$$dx(t) = A(r(t))x(t)dt + \sum_{k=1}^m B_k(r(t))x(t)dw_k(t) \tag{4}$$

on $t \ge 0$, with initial data $x(0) = x_0 \in L^2_{\mathcal{F}_0}(R^n)$. Here $A,\ B_k : S \to R^{n\times n}$ and we will often write $A(i) = A_i$ and $B_k(i) = B_{ki}$. Suppose that this given equation is unstable and we are required to design a feedback control $u(x(\delta(t)), r(t))$ based on the discrete-time state observations in the drift part so that the controlled SDE

$$dx(t) = [A(r(t))x(t) + u(x(\delta(t)), r(t))]dt + \sum_{k=1}^m B_k(r(t))x(t)dw_k(t) \tag{5}$$

will be mean-square exponentially stable, where u is a mapping from $R^n \times S$ to R^n, $\tau > 0$ and

$$\delta(t) = [t/\tau]\tau \quad \text{for } t \ge 0, \tag{6}$$

in which $[t/\tau]$ is the integer part of t/τ. We repeat that the feedback control $u(x(\delta(t)), r(t))$ is designed based on the discrete-time state observations $x(0), x(\tau), x(2\tau), \dots$, though the given hybrid SDE (4) is of continuous time. As the given SDE (4) is linear, it is natural to use a linear feedback control. One of the most common linear feedback controls is the structure control of the form $u(x, i) = F(i)G(i)x$, where F and G are mappings from S to $R^{n\times l}$ and $R^{l\times n}$, respectively, and one of them is given while the other needs to be designed. These two cases are known as:

- State feedback: design $F(\cdot)$ when $G(\cdot)$ is given.
- Output injection: design $G(\cdot)$ when $F(\cdot)$ is given.

Again, we will often write $F(i) = F_i$ and $G(i) = G_i$. As a result, controlled system (5) becomes

$$dx(t) = [A(r(t))x(t) + F(r(t))G(r(t))x(\delta(t))]dt + \sum_{k=1}^m B_k(r(t))x(t)dw_k(t). \tag{7}$$

We observe that Eq. (7) is in fact a stochastic differential delay equation (SDDE) with a bounded variable delay. Indeed, if we define the bounded variable delay $\zeta : [0, \infty) \to [0, \tau]$ by

$$\zeta(t) = t - v\tau \quad \text{for } v\tau \le t < t(v+1)\tau, \tag{8}$$

and $v = 0, 1, 2, \dots$, then Eq. (7) can be written as

$$dx(t) = [A(r(t))x(t) + F(r(t))G(r(t))x(t-\zeta(t))]dt + \sum_{k=1}^m B_k(r(t))x(t)dw_k(t). \tag{9}$$

It is therefore known (see e.g. [22]) that Eq. (7) has a unique solution $x(t)$ such that $\mathbb{E}|x(t)|^2 < \infty$ for all $t \ge 0$.

3. Main results

In this section, we will first write $F(r(t))G(r(t)) = D(r(t))$ and establish the stability theory for the following hybrid SDE

$$dx(t) = [A(r(t))x(t) + D(r(t))x(\delta(t))]dt + \sum_{k=1}^m B_k(r(t))x(t)dw_k(t). \tag{10}$$

We will then design either $G(\cdot)$ given $F(\cdot)$ or $F(\cdot)$ given $G(\cdot)$ in order for controlled SDE (7) to be stable.

3.1. Stability of SDE (10)

Let us begin with a useful lemma.

Lemma 3.1. *Set*

$$M_A = \max_{i\in S} \|A_i\|^2, \qquad M_D = \max_{i\in S} \|D_i\|^2,$$

$$M_B = \max_{i\in S} \sum_{k=1}^m \|B_{ki}\|^2,$$