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Research paper

Hyperbolic relative equilibria for the negative curved n–body problem☆

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ABSTRACT

We consider the n–body problem defined on surfaces of constant negative curvature. For the five and seven body case with a symmetric configuration, we give conditions on initial positions for the existence of collinear hyperbolic relative equilibria, where here collinear means that the bodies are on the same geodesic. If they satisfy the above conditions, then we give explicitly the values of the masses in terms of the positions, such that they can lead to these type of relative equilibria. The set of parameters that lead to these type of solutions has positive Lebesgue measure. For the general case of n–equal masses, we prove the no-existence of hyperbolic relative equilibria with a regular polygonal shape. In particular the Lagrangian hyperbolic relative equilibria do not exist.

1. Introduction

We consider the gravitational n–body problem in the two-dimensional hyperbolic space. This interesting problem has its origins in the ideas proposed by Lovachevski in the 1830s about non-Euclidean geometries [12]. We use the formulation of the problem proposed by Diacu, Pérez–Chavela and Santoprete which can be found in [7,8]. For an interesting historical background about this problem you can see [4].

Relative equilibria are solutions of the equations of motion where the distances among the particles remain constant along the time. These solutions have been widely studied in the classical problem since the Euler and Lagrange times. In the last years you can find some results on this subject in curved spaces, in this way, the classical Newtonian Eulerian and Lagrangian solutions has been generalized to spaces of constant curvature in several papers, see for instance [6,7,17]. Some results about the stability of these solutions can be found in [9,11], where we can see some differences with respect to the Newtonian case. An interesting point to emphasize is that, in the positively-curved problem for $n = 3$, the linear stability of the Lagrangian solutions depends on the angular momentum [11]. The Kepler problem on spaces of constant curvature has been tackled in several papers, as for instance [1,3]. We say that a solution of the n–body problem on these spaces is collinear, if all masses are on the same geodesic for all time. In this way, for three masses in a collinear relative equilibria (particles in a rotating geodesic on S^2) we have that the stability depends on positions and masses [15]. Some authors have also studied the polygonal relative equilibria in the positive curved space [5,16]. We can see that as in the classical Newtonian n–body problem (zero curvature), n–equal masses located at the vertices of a regular n–gon generate a relative

☆ To Florin Diacu, our friend and teacher, in memoriam.

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equilibria by taking the action of the $SO(2)$–group. In particular for $n = 3$ and positive curvature, it has been shown in [7], that three masses form a Lagrangian relative equilibrium (three equal distances) iff they are equal

In the last years, the literature has been increased with many papers about relative equilibria and their stability in spaces of positive curvature, as the mentioned above, however very few on this subject for the negative curvature case have been studied. One of them is the paper written by Tibboel [16], where he shows the non existence of elliptic homographic orbits for irregular polygons with coordinate z not constant (he considers positive and negative curvature). Since, up to isometries, there are three different groups of isometry in surfaces of constant negative curvature, in principle we may have three different classes of relative equilibria called elliptic, hyperbolic or parabolic (ahead in this paper we will go deeper in this point), however it is known that the parabolic relative equilibria do not exist [7].

The main results presented in this paper are related with symmetric collinear-hyperbolic relative equilibria in the 5 and 7–body problem and on polygonal relative equilibria on spaces of negative curvature. In the first case we consider a symmetric configuration in H^2 and we face the problem about the distribution of positions and masses in order to get a relative equilibrium. We show that the set of parameters corresponding to positions and masses which lead to collinear hyperbolic relative equilibria has positive Lebesgue measure. In both cases, 5 and 7–bodies in a symmetric collinear-hyperbolic relative, the mass in the middle is arbitrary, so our results extend to the 4 and 6–bodies in a collinear-hyperbolic relative equilibria by taking the middle mass equal to zero. Until we know, these kind of results and the techniques that we have used to prove them have not been reported before.

Another results in this paper are related with polygonal hyperbolic relative equilibria, we prove that these kind of solutions do not exist, in particular there are not Lagrangian hyperbolic relative equilibria.

After the introduction and the preliminaries studied in Section 2, where we present two models for the surfaces of constant negative curvature in Section 3 we analyze the collinear relative equilibria of five and seven bodies. Our main result prove the existence of new families of collinear hyperbolic relative equilibria for the 5–body problem on spaces of negative curvature. We present also a result related to 7 particles and the existence of positions and masses such that they lead to hyperbolic relative equilibria, these results and the techniques that we have used to proved them are totally originals. The above results show the existence of new families of hyperbolic relative equilibria for the negative curved space which come to enrich the knowledge of dynamics in these spaces. In Section 4 we prove the non-existence of hyperbolic relative equilibria with polygonal regular n–gon shape for the n–body problem with equal masses on spaces of negative curvature, in particular for $n = 3$ this shows the no-existence of hyperbolic Lagrangian relative equilibria. Actually we proved first the last result, then we discovered that our result could extend for the general n–gon with equal masses at their vertices.

2. Relative equilibria on surfaces of negative curvature

We consider the two dimensional space of constant curvature -1, $\mathcal{H}^2$. There are several models from the hyperbolic geometry (all of them isometric) to represent $\mathcal{H}^2$. In this paper we will use two of them, the Weierstrass model denoted by $\mathbb{L}^2$ and the Poincaré upper half plane $\mathbb{H}^2$.

2.1. The Weierstrass model

The Weierstrass model, also known as the pseudo sphere $\mathbb{L}^2$, is given by the upper sheet of the hyperboloid

$$\mathbb{L}^2 = \{(x, y, z) \in \mathbb{R}^{2,1} | x^2 + y^2 - z^2 = -1\},$$

where $\mathbb{R}^{2,1}$ is the Minkowski space, that is $\mathbb{R}^3$ endowed with the Lorentz inner product denoted by $\odot$ (given $a = (a_x, a_y, a_z)$ and $b = (b_x, b_y, b_z)$, we have $a \odot b = a_x b_x + a_y b_y - a_z b_z$)). We denote by q_i the position of the particle with mass m_i. The distance between any two points in this space is given by $d(q_i, q_j) = \cosh^{-1}(-q_i \odot q_j)$. The potential is given by

$$U(q) = \sum_{i<j} m_i m_j \cot(d(q_i, q_j)).$$

And the kinetic energy is defined by

$$T = \frac{1}{2} \sum_i m_i \dot{q}_i \odot \dot{q}_i.$$

From the Euler-Lagrange equations, the equations of motion take the form

$$\ddot{q}_i = \sum_{i \neq j} \frac{m_j (q_j + (q_i \odot q_j) q_i)}{(-1 + (q_i \odot q_j)^2)^{3/2}} + (\dot{q}_i \odot \dot{q}_i) q_i, \quad i = 1, \cdots, n. \tag{1}$$

Let $Lor(\mathbb{L}^2, \odot)$ be the group of all orthogonal transformations of determinant 1 that maintains the upper part of the hyperboloid invariant (the group of isometries of $\mathbb{L}^2$), see [7,14] for more details. Applying the Principal Axis Theorem [13], which states that any 1−parameter subgroup of $Lor(\mathbb{L}^2, \odot)$ can be written, in a proper basis, as

$$A = P \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1},$$