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Research paper

Exact solution of the Zakharov–Shabat scattering problem for doubly-truncated multisoliton potentials

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ABSTRACT

Recent studies have revealed that multisoliton solutions of the nonlinear Schrödinger equation, as carriers of information, offer a promising solution to the problem of nonlinear signal distortions in fiber optic channels. In any nonlinear Fourier transform based transmission methodology seeking to modulate the discrete spectrum of the multisolitons, choice of an appropriate windowing function is an important design issue on account of the unbounded support of such signals. Here, we consider the rectangle function as the windowing function for the multisolitonic signal and provide a recipe for computing the exact solution of the associated Zakharov–Shabat (ZS) scattering problem for the windowed/doubly-truncated multisoliton potential. The idea consists in expressing the Jost solution of the doubly-truncated multisoliton potential in terms of the Jost solution of the original potential. The proposed method allows us to avoid prohibitive numerical computations normally required in order to accurately quantify the effect of time-domain windowing on the nonlinear Fourier spectrum of the multisolitonic signals. Further, the method devised in this work also applies to general type of signals admissible as ZS scattering potential, and, may prove to be a useful tool in the theoretical analysis of such systems.

Notations

The set of non-zero positive real numbers ($\mathbb{R}$) is denoted by $\mathbb{R}_+$. For any complex number ζ, $\mathrm{Re}(\zeta)$ and $\mathrm{Im}(\zeta)$ refer to the real and the imaginary parts of ζ, respectively. The complex conjugate of ζ is denoted by ζ^*. The upper-half (lower-half) of complex plane ($\mathbb{C}$) is denoted by $\mathbb{C}_+$ ($\mathbb{C}_-$) and its closure by $\overline{\mathbb{C}}_+$ ($\overline{\mathbb{C}}_-$). The Pauli's spin matrices are denoted by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For the sake of uniformity of notation, we set $\sigma_0 = \mathrm{diag}(1, 1)$. The support of a function $f : \Omega \to \mathbb{R}$ in Ω is defined as $\mathrm{supp}\, f = \overline{\{x \in \Omega |\, f(x) \neq 0\}}$. The Lebesgue spaces of complex-valued functions defined in $\mathbb{R}$ are denoted by L^p for $1 \leq p \leq \infty$ with their corresponding norm denoted by $\|\cdot\|_{\mathsf{L}^p}$ or $\|\cdot\|_p$.

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1. Introduction

In optical fiber communication, the propagation of optical field in a loss-less single mode fiber under Kerr-type focusing nonlinearity is governed by the nonlinear Schrödinger equation (NSE) [1,2] which, in its standard form, reads as

$$i\partial_x q = \partial_t^2 q + 2|q|^2 q, \quad (t, x) \in \mathbb{R} \times \mathbb{R}_+, \tag{1}$$

where $q(t, x)$ is a complex valued function associated with the slowly varying envelope of the electric field, t is the retarded time and x is position along the fiber. This equation also provides a satisfactory description of optical pulse propagation in the guiding-center or path-averaged formulation [3–5] when more general scenarios such as presence of fiber losses, lumped or distributed periodic amplification are included in the mathematical model of the physical channel.

The initial value problem (IVP) corresponding to the NSE was first solved by Zakharov and Shabat [6], which is known to be one of the first successful implementations of the *inverse scattering transform* (IST) method. *Multisolitons* or, more precisely, *K-soliton* solutions were obtained as a special case of this theory. The IST method was later extended to a wider class of nonlinear evolution equations known as the Ablowitz–Kaup–Newell–Segur (AKNS) class of integrable equations [7,8]. In this pioneering work, IST was, for the first time, presented as a way of Fourier analysis for nonlinear problems prompting researchers to coin the term *nonlinear Fourier transform* (NFT) for IST. In this terminology, any subset of the scattering data that qualifies as the "primordial" scattering data [8] is referred to as the *nonlinear Fourier spectrum*.

The fact that the energy content of K-soliton solutions is not dispersed away as it propagates along the fiber makes them promising as carriers of information in optical communication. These ideas were first explored by Hasegawa and Nyu [9] who proposed encoding information in the eigenvalues of the K-soliton solutions in a framework which they described as the *eigenvalue communication*. With the recent breakthroughs in coherent optical communication [10,11] and growing need for increased channel capacity [12–14], these ideas have been recently revived. We refer the reader to the comprehensive review article [15] and the references therein for an overview of NFT-based optical communication methodologies and its potential advantage over the conventional ones.

In this article, we focus on a particular aspect of the NFT-based transmission methodologies which seek to modulate the discrete part of the nonlinear Fourier spectrum using K-solitons as information carriers. Given that the support of the K-soliton solutions is infinite, it is mandatory to employ a windowing function [16]. The windowing function must be such that it does not considerably alter the nonlinear Fourier spectrum of the original signal. In this work, we consider the simplest of the windowing functions, the *rectangle function*. It is shown that the resulting scattering problem for the "windowed" or the doubly-truncated K-soliton solution is exactly solvable. The idea is to express the Jost solutions of the windowed potential in terms of the Jost solutions of the original potential. Such an approach has already appeared in the work of Lamb [17] where the scattering problem for a potential truncated from one side is solved exactly using the Jost solutions of the original potential. In particular, the observation that truncated K-soliton solution has rational reflection coefficient has been used to devise exact techniques for IST [18–20]. Adapting Lamb's approach, it is further shown that, in the case of truncation from both sides, one can set up a Riemann–Hilbert (RH) problem to obtain the Jost solutions of the doubly-truncated potential. It must be noted that this method applies to general potentials; however, for K-soliton solutions, the evaluation of certain integrals become a trivial task and the solution of the RH problem can be obtained in a closed form. In particular, the method of *Darboux transformation* (DT) for computing K-soliton solutions provide an adequate representation of the Jost solutions in terms of the so called *Darboux matrix* which, as a function of the spectral parameter, has a rational structure facilitating the solution of the RH problem. This representation further enables us to obtain precise estimates for the effective temporal support as well as spectral width of the K-soliton pulses. The rational structure of the aforementioned Darboux matrix has also been recently exploited to develop fast numerical algorithms for DT [21] and IST [22].

2. Direct scattering: doubly-truncated potential

The NFT of a given complex-valued signal $q(t)$ is introduced via the associated *Zakharov–Shabat scattering problem* (or ZS problem in short) [6] which can be stated as follows: Let $\zeta \in \mathbb{R}$ and $\boldsymbol{v} = (v_1, v_2)^\intercal \in \mathbb{C}^2$, then

$$\boldsymbol{v}_t = -i\zeta\sigma_3\boldsymbol{v} + U\boldsymbol{v}, \tag{2}$$

where the matrix elements of U are $U_{11} = U_{22} = 0$ and $U_{12} = q(t) = -U_{21}^* = -r^*(t)$. Here, $q(t)$ is identified as the *scattering potential*. Henceforth, we closely follow the formalism developed in [8,23]. We assume that the Jost solutions of the *first kind*, denoted by $\boldsymbol{\psi}(t;\,\zeta)$ and $\overline{\boldsymbol{\psi}}(t;\zeta)$, which are the linearly independent solutions of (2), are known. These solutions are characterized by the following asymptotic behavior as $t\to\infty$: $\boldsymbol{\psi}(t;\zeta)e^{-i\zeta t} \to (0,1)^\intercal$ and $\overline{\boldsymbol{\psi}}(t;\zeta)e^{i\zeta t} \to (1,0)^\intercal$. We also assume that the Jost solutions of the *second kind*, denoted by $\boldsymbol{\phi}(t,\zeta)$ and $\overline{\boldsymbol{\phi}}(t,\zeta)$, which are also linearly independent solutions of (2) are known. These solutions are characterized by the following asymptotic behavior as $t \to -\infty$: $\boldsymbol{\phi}(t;\,\zeta)e^{i\zeta t} \to (1,\,0)^\intercal$ and $\overline{\boldsymbol{\phi}}(t;\zeta)e^{-i\zeta t} \to (0,-1)^\intercal$. The scattering coefficients corresponding to $q(t)$ can be written in terms of the Jost solutions by using the Wronskian relations [8]

$$\begin{aligned} a(\zeta) &= \mathscr{W}\left(\boldsymbol{\phi},\boldsymbol{\psi}\right), \quad b(\zeta) = \mathscr{W}\left(\overline{\boldsymbol{\psi}},\boldsymbol{\phi}\right),\\ \bar{a}(\zeta) &= \mathscr{W}\left(\overline{\boldsymbol{\phi}},\overline{\boldsymbol{\psi}}\right), \quad \bar{b}(\zeta) = \mathscr{W}\left(\overline{\boldsymbol{\phi}},\boldsymbol{\psi}\right). \end{aligned} \tag{3}$$