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Liveness Analysis of Petri Nets Using Siphons and Mathematical Programming ⋆

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Abstract: Liveness is a significant behavioral property of a Petri net. Siphons, as structural objects, are closely related to the liveness of a Petri net. For generalized systems of simple sequential processes with resources (GS^{3}PR), a mixed integer programming (MIP) model is formulated, which can detect the existence of minimal non-max*-marked siphons that cause deadlocks or livelocks. We conclude that a GS^{3}PR is live if there is no feasible solution to the formulated MIP model. An example is used to illustrate the proposed method.

Keywords: Petri net, liveness, deadlock, controllability, siphon, mixed integer programming (MIP).

1. INTRODUCTION

Petri nets (Li and Zhou, 2009; Li et al., 2012) are an edged mathematical tool for modeling and control of discrete event systems. Reachability graph analysis (Uzam, 2002; Chen and Li, 2011; Chen et al., 2011) and structural analysis (Li and Zhou, 2004; Li and Zhao, 2008; Liu et al., 2013c, 2010, 2013a,b) are two major methods to deal with deadlocks in a system modeled with Petri nets. The former considers the complete reachable state space of a Petri net model, whose computation is prohibitive, and thus limits its application to real-world systems. The latter receives more attention thanks to the computational performance.

Siphons are closely related to the liveness of Petri nets (Barkaoui and Pradat-Peyre, 1996). It is important to find them that may cause deadlocks. However, the performance of siphon-based liveness-enforcing approaches is degraded and deteriorated as their number grows quickly beyond practical limits and in the worst case grows exponentially fast with respect to the Petri net size. Hence, an efficient enumeration of problematic siphons is of particular interest. Recently, MIP approaches are employed to find problematic siphons. They can be used to verify some properties of structurally bounded Petri net models. Although an MIP problem is NP-hard in theory, the mathematical programming nature, independent from the initial marking of a net, opens a new way of checking liveness of Petri nets.

Chu and Xie (1997) first use MIP to detect whether a structurally bounded Petri net is deadlock-free. This method avoids an explicit enumeration of all strict minimal siphons (SMS). However, it is only suitable for ordinary Petri nets. Detection of problematic siphons in a generalized Petri net by MIPs is more complicated.

In Zhao et al. (2010), based on deadly marked siphons (DMS) in well-marked systems of sequential systems with shared resources (S^{4}R), Zhao *et al.* modify the MIP test in Chu and Xie (1997) to detect DMS for S^{4}R. However, an S^{4}R may have livelocks even though it is deadlock-free. In this case, the siphons causing livelocks cannot be detected by the modified MIP and the net cannot be further controlled. Furthermore, the techniques in both Chu and Xie (1997) and Zhao et al. (2010) cannot obtain a minimal problematic siphon directly.

Zhong and Li (2009) propose an MIP model to detect a minimal non-max-marked siphon. However, their method cannot detect the siphons that cause livelocks. Furthermore, it outputs an SMS when a Petri net is live with non-max-marked siphons, creating a false impression that the net is non-live and thus needs a control place to control it.

In Liu and Li (2010), we improve the MIP-based methods in the literature in terms of the max$''$-controllability condition of siphons. We define extended DMS (EDMS) and then develop a more general MIP model that can detect deadlocks and livelocks caused by siphons in an S^{4}R. We conclude that the net is live if there is no feasible solution for the MIP model. This programming is more powerful than the MIPs in Zhao et al. (2010) and Zhong and Li (2009) but still restrictive since it outputs an SMS when a Petri net is live with non-max$''$-marked siphons.

Based on the max*-controllability condition of siphons, this study, for GS^{3}PR (Liu and Barkaoui, 2013), proposes

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a new MIP model that can detect minimal problematic siphons directly. We conclude that if there is no feasible solution to this model, the net is live. Since the approach is based on siphons and mathematical programming, its computational efficiency is relatively insensitive to the initial marking. The basics of Petri nets and GS^{3}PR can be referred to Li and Zhou (2009), Liu and Li (2010), and Liu and Barkaoui (2013).

2. LIVENESS DETECTION FOR GS^{3}PR BASED ON MAX*-CONTROLLABILITY CONDITION

Without loss of generality, in what follows, (N, M_0) with $N = (P_A \cup P^0 \cup P_R, T, F, W)$ denotes a GS^{3}PR, where P_A, P^0, and P_R are the sets of the operation, process idle, and resource places, respectively. T is the set of transitions with $(P_A \cup P^0 \cup P_R) \cap T = \emptyset$ and $(P_A \cup P^0 \cup P_R) \cup T \neq \emptyset$. F is called a flow relation of the net, represented by arcs with arrows from places to transitions or from transitions to places. W is a mapping that assigns a weight to an arc: $W(f) > 0$ if $f \in F$ and $W(f) = 0$ otherwise. For an arc $(p, t) \in F$, e_{pt}, a binary variable, indicates whether the arc is enabled (enabled: $M(p) \geq W(p, t) \Rightarrow e_{pt} = 1$; disabled: $M(p) < W(p, t) \Rightarrow e_{pt} = 0$). S is a siphon in a GS^{3}PR with $S = S_A \cup S_R$, $S_R = S \cap P_R$, and $S_A = S \backslash S_R$. The complementary set of S, denoted by $[S]$, is a set of operation places that are not in S but their execution needs the resources contained in S. $T_r^c = r^\bullet \cap [S]^\bullet$ denotes the set of critical transitions of r, where $r \in S_R$. $T_S^c = S_R^\bullet \cap [S]^\bullet$ denotes the set of critical transitions of S. The reachability set of (N, M_0) is denoted as $R(N, M_0)$. $R_S(N, M_0) = \{M | M = M_0 + [N]Y, M \geq 0, Y \geq 0\}$ is the marking set decided by the state equation of a net (N, M_0).

Definition 1. (Liu and Barkaoui, 2013) Let S be a strict siphon in a well-marked GS^{3}PR net (N, M_0). S is said to be max*-marked (non-max*-marked) at $M \in R(N, M_0)$ if at least one (none) of the following conditions holds:

(i) $\exists p \in S_A$, $M(p) \geq 1$;

(ii) $\exists r \in S_R$, $M(r) \geq max_{t \in T_r^c} W(r, t)$;

(iii) $\exists t \in T_S^c$, $e_{pt} = 1$ and $e_{rt} = 1$ (t is enabled at M).

Definition 2. (Liu and Barkaoui, 2013) Let S be a strict siphon in a well-marked GS^{3}PR net (N, M_0). S is said to be max*-controlled if S is max*-marked at any reachable marking from M_0.

Theorem 3. (Liu and Barkaoui, 2013) Let (N, M_0) be a well-marked GS^{3}PR net and $\Pi \neq \emptyset$ be the set of SMS. It is live iff $\forall S \in \Pi$, S is max*-controlled.

An immediate implication of this theorem is that a minimal non-max*-marked siphon at a marking $M \in R(N, M_0)$ can be determined by the following integer programming problem if the net is non-live.

Theorem 4. Let (N, M_0) be a well-marked GS^{3}PR net system. A minimal non-max*-marked siphon S and a corresponding marking $M \in R_S(N, M_0)$ can be obtained through the following MIP formulation:

$$min \sum_{p \in P \backslash P^0} s_p \tag{1}$$

subject to

For all $t \in T$, $p \in P$:

$$|t^\bullet| \sum_{p \in {}^\bullet t} s_p \geq \sum_{p \in t^\bullet} s_p \tag{2}$$

$$\sum_{p \in P^0} s_p = 0 \tag{3}$$

$$\sum_{p \in P_A} s_p \geq 1 \tag{4}$$

$$\sum_{p \in P_R} s_p \geq 2 \tag{5}$$

For all $p \in P_A, t \in p^\bullet$

$$e_{pt} \geq \frac{M(p)}{\psi(p)} \tag{6}$$

$$M(p) \geq e_{pt} \tag{7}$$

$$s_p + e_{pt} \leq 1 \tag{8}$$

$$\sum_{t \in T \backslash P^{0\bullet}} e_{pt} \leq |\{t \in T \backslash P^{0\bullet}\}| - 1 \tag{9}$$

For all $r \in P_R, t \in r^\bullet$

$$e_{rt} \geq \frac{M(r) - W(r, t) + 1}{M_0(r) - W(r, t) + 1} \tag{10}$$

$$\frac{M(r)}{W(r, t)} \geq e_{rt} \tag{11}$$

$$\sum e_{rt} + s_r \leq |r^\bullet| \tag{12}$$

For all $r, r' \in P_R, t \in r'^\bullet, r \in t^\bullet \cap P_R, p \in^\bullet t \cap P_A$

$$(2s_{r'} - 1) \cdot M(r') \leq (2s_{r'} - 1) \cdot \{max[s_{r'} \cdot s_r \cdot W(r', t)] - s_{r'}\} \tag{13}$$

$$e_{r't} \cdot e_{pt} \cdot s_{r'} = 0 \tag{14}$$

$$s_p, e_{rt}, e_{pt} \in \{0, 1\} \tag{15}$$

$$M = M_0 + [N]Y, M \geq 0, Y \geq 0 \tag{16}$$

where $\psi(p)$ can be directly obtained from the definition of a GS^{3}PR.

The minimal non-max*-marked siphon is the set of places whose associated variables s_p's are 1.

Proof. Let us first make some comments on the variables used in the constraints.

Constraints (2)–(5): Constraint (2) ensures that s is the characteristic vector of siphon S Liu and Li (2010). Constraints (3)–(5) guarantee that the solution obtained contains no idle place, at least an operation place and two resource places Liu and Barkaoui (2013).

Constraints (6)–(9): For each $t \in p^\bullet$, $p \in P_A$, e_{pt} indicates whether arc (p, t) is enabled. It follows immediately from the following facts:

- Since $\psi(p) > 0$, $M(p)/\psi(p) > 0$ if $M(p) > 0$, which is equivalent to $e_{pt} = 1$.
- $e_{pt} = 0$ if $M(p) = 0$.