12th IFAC/IEEE Workshop on Discrete Event Systems
Cachan, France. May 14-16, 2014

A Series Expansion Approach to Risk Analysis of an Inventory System with Sourcing

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Abstract: In this paper we extend the series expansion approach for uni-chain Markov processes to a special case of finite multi-chains with possible transient states. We will show that multi-chain Markov models arise naturally in simple models such as a single item inventory system with sourcing, i.e., with the possibility to choose between different suppliers for replenishment. Numerical examples are provided to illustrate the performance of the series expansion algorithm to the risk analysis of this type of inventory systems.

Keywords: sensitivity analysis, perturbation analysis for Markov chains, multi-chains, series expansion algorithm

1. INTRODUCTION

Consider a Markov process $X_\theta = \{X_\theta(t) : t \geq 0\}$ on some finite state-space S, where θ denotes a parameter of the system, such as the service rate in a queue or the arrival rate to a network. Let Q_θ denote the infinitesimal generator of X_θ. The transition probability from $X_\theta(0) = i$ to $X_\theta(t) = j$, for $t > 0$, is denoted by $P_\theta(i, j; t)$, for $i, j \in S$ and it holds in matrix form $P_\theta(t) = e^{Q_\theta t}$, $t \geq 0$. Denote, for $i, j \in S$, by

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t P_\theta(i, j; u)\, du = \Pi_\theta(i, j)$$

the ergodic projector of Q_θ. For any initial distribution μ on S it then holds that the limiting distribution of the process is $\mu\Pi_\theta$. A Markov process is called uni-chain if there exists only one ergodic class, and multi-chain otherwise. Provided that X_θ is uni-chain with possible transient states, then the stationary distribution π_θ of X_θ exists and can be found as the unique probability vector solving $\pi_\theta Q_\theta = 0$. In this case, Π_θ is the matrix with rows equal to π_θ.

Perturbation analysis of uni-chain Markov processes is a well developed theory that studies the effect of a perturbation Δ of θ has on π_θ. For an early paper see Schweitzer [1968]. The series expansion algorithm introduced in Heidergott and Hordijk [2003], Heidergott et al. [2007, 2010] allows to obtain $\pi_{\theta+\Delta}$ as a polynomial in Δ and thus allows for a functional approximation of π_θ on an entire interval. See also Cao [1998] and Abbas et al. [2013].

This paper is devoted to perturbation analysis of multi-chains. We will show that the series expansion algorithm can be extended to a special multi-chain case, with possible transient states, provided that Q_θ and $Q_{\theta+\Delta}$ either have the same ergodic classes with some restrictions on the transient states of $Q_{\theta+\Delta}$, or Q_θ is uni-chain. In addition we will show that, without extending the algorithm, it is not possible to develop $\pi_{\theta+\Delta}$ into a series at θ if Δ has an influence on the ergodic classes.

We will apply multi-chain version of the series expansion algorithm to perturbation analysis of a single-item inventory system with sourcing. Customers arrive to the inventory according to a Poisson λ process and purchase one single time at a time. The inventory can be replenished by either ordering from a wholesaler, with lead time μ_1, or directly from the producer, with lead time μ_2 (throughout this paper 1 and 2 will be used to indicate orders from the wholesaler and from the producer, respectively). From the wholesaler items can be ordered in single units, whereas for the producer a bulk order has to be placed, i.e., the product can only be ordered in units of size $b > 1$. When the inventory level drops below s_2 a bulk order is placed. Since, the lead time of a bulk order is relatively large, it can become necessary to order single items if the inventory reaches level $s_1 > 0$. The maximal capacity of the inventory is denoted by $N_{\max}$. If at inventory level s_2 a bulk order is placed, then maximal number of batches to be ordered is given by

$$\left\lfloor \frac{N_{\max} - s_2}{b} \right\rfloor,$$

and the inventory is replenished to level

$$S_2 := \left\lfloor \frac{N_{\max} - s_2}{b} \right\rfloor b.$$

If, single items are ordered, then we assume that the inventory is replenished to level $S_1 \leq N_{\max}$. Thus, the system is described through (s_1, S_1, s_2, S_2). To summarize, the system is specified by the set of *uncontrollable* parameters λ, μ_1, μ_2, and the set of *controllable* parameters (s_1, S_1, s_2, S_2).

As we will illustrate by examples, the Markov processes modeling the inventory level together with the order list (to be defined presently) is typically a multi-chain with transient states. From our analysis of the multi-chain case, we will conclude that we cannot use the series expansion algorithm for analyzing the dependence of the ergodic projector on $\theta = (s_1, S_1, s_2, S_2)$. For this reason we focus in this paper on a series expansion of the ergodic projector Π_θ with respect to $\theta = \lambda, \mu_1, \mu_2$. More specifically, we will investigate the robustness of our model against the statistical insecurity in the parameter values for θ. To this end we make the reasonable assumption that the "true" lead time μ_1 is not revealed to us and for the model we have to rely on statistics for estimating μ_1. For example letting $\theta = \mu_1$ we let $\theta = \theta_0 + \Delta$ with θ_0 being the mean value of the statistics and Δ representing the insecurity in predicting μ_1 (typically, Δ is normally distributed with mean zero). Alternatively, Δ may be obtained by expert judgment, like, Δ uniformly distributed on $[-\epsilon, \epsilon]$ for some dispersion measure $\epsilon > 0$.

The main idea of our analysis is to develop $\Pi_{\theta=\theta_0+\Delta}$ into a Taylor series with respect to Δ at $\Delta = 0$, i.e., we will have

$$\Pi_{\theta=\theta_0+\Delta} \approx H(\theta_0, N; \Delta) := \Pi_{\theta_0} \sum_{n=0}^{N} ((Q_{\theta_0+\Delta} - Q_{\theta_0}) D_{\theta_0})^n, \qquad (1)$$

where D_θ is the deviation matrix associated with Q_θ and is given by

$$D_\theta(i,j) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \big(P_\theta(i,j,u) - \Pi_\theta(i,j)\big) du,$$

for $i, j \in S$. For computation, the deviation matrix D_θ can be obtained as

$$D_\theta = (\Pi_\theta - Q_\theta)^{-1} - \Pi_\theta,$$

assuming that $||I + Q_\theta - \Pi_\theta|| < 1$ for some operator norm $||\cdot||$ (which follows from the results on Neumann series). The series expansion in (1) allows to write Π_θ as a polynomial in the random variable Δ. We will use this property to study the behavior of Π_θ as a random variable induced by Δ. This provides a powerful analysis of the risk introduced by the insecurity about the actual value of θ on the model. Note that $\mathbb{E}[\theta] = \theta_0$, but, as Π_θ typically depends in a non-linear way on Δ, it does not hold true that $\Pi_{\theta_0} = \mathbb{E}[\Pi_\theta]$.

The paper is organized as follows. In Section 2 we discuss the series expansion algorithm for a special case of finite multi-chain Markov processes. The risk analysis with respect to Δ is detailed in Section 3. Finally, in Section 4 we introduce the inventory model in more detail and we present numerical results.

2. SERIES EXPANSION ALGORITHM FOR MULTI-CHAINS

Let Q have I ergodic classes and T transient states. After appropriate relabeling of the states, Π_Q can be written as

$$\begin{pmatrix} \Pi_1 & 0 & \dots & & 0 \\ 0 & \Pi_2 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & \Pi_I & 0 \\ R_1 & \dots & & R_I & 0 \end{pmatrix}, \qquad (2)$$

where Π_i is the ergodic projector of the i-th ergodic class, which is given by a matrix with rows equal to π^i, the uniquely defined stationary distribution on class i, and $R_i(j,k)$ is the probability to go from transient state j to state k (which lies in ergodic class i). Note that it is typically possible to go from a transient state to several ergodic classes. In this case $R_i(i,k)$ is the probability of going from state i to ergodic class i times $\pi^i(k)$.

Remark 1. Computing the ergodic projector of a multi-chain with transient states is a non-trivial task. For computations, we apply the idea in E. P. C. Kao [1997] on page 172 for identifying the irreducible classes and transient states. Once the irreducible classes are identified, the corresponding stationary distributions Π_i are computed in the standard way, i.e., we find the probability solution of $\pi_i Q_i = 0$, where Q_i is the restriction of the Q matrix to states from ergodic class i, and Π_i is a matrix with rows equal to π_i. The values of the transient components R_i requires solving a set of balance equations. In particular, define the $T \times I$ matrix P_{T_1} as the one step probabilities of the embedded Markov chain of going from a transient state to an ergodic class. More specifically, $P_{T_1}(i,c)$ is the one step probability of going from transient state i to the ergodic class c. Similar, define the $T \times T$ matrix P_{T_2} as the one step probabilities of moving from one transient state to another one. Then the probabilities of ending in ergodic class c while starting in transient state i is given by the (i,c)-th element in the $(T \times I)$-matrix $(I_T - P_{T_2})^{-1} P_{T_1}$, where I_T is the $T \times T$ identity matrix. The rows in the ergodic projector that correspond with the transient states can then be calculated using these probabilities and Π_i.

We say that *a Markov process X (resp. a infinitesimal generator Q) dominates a Markov process X' (resp. an infinitesimal generator Q')* if X and X' are defined on a common state space and (i) the ergodic classes of X' are subsets of the ergodic classes of X, and (ii) it must hold for any transient state in Q' that the set of ergodic class(es) that can be reached with a positive probability is a subset of the only ergodic class that can be reached from the same state in Q (observe that the transient state in Q' is not necessarily transient in Q). Taking the explicit form in (2) into consideration we can proof the following result in case of domination.

Lemma 2. If Q dominates Q', then $\Pi_{Q'}\Pi_Q = \Pi_Q$.

Proof: In order to rephrase the domination definition above in mathematical terms and to prove the lemma we need some definitions. Define $E(Q)$ as the set of ergodic states of Q, similar, define $T(Q)$ as the set of transient states of Q. More specifically, define $[i]_Q$ as the set of states belonging to the ergodic class in Q which contains $i \in E(Q)$. Furthermore, define $TE(Q;i)$ as the ergodic states in Q which have a positive probability of being reached from state i. Using this notation, the domination definition satisfies:
(i) $\forall i \in E(Q') : [i]_{Q'} \subseteq [i]_Q$.