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Short communication

Sliding motion on the intersection of two manifolds: Spirally attractive case ☆

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ABSTRACT

In this note, we consider sliding motion on the intersection Σ of two smooth manifolds in the case when the dynamics near the manifold Σ is spiral-like, and Σ is spirally attractive. We clarify the meaning of spiral-like dynamics around Σ, characterize what we mean by spiral attractivity of Σ, and finally discuss what to expect when Σ ceases to be attractive, with nearby orbits getting farther away from Σ through spiraling motion. Our characterization of spiral-attractivity of Σ is given by a single number, which plays a role similar to that of a *Floquet multiplier* for a smooth planar system.

1. Introduction

Piecewise smooth systems (differential systems with discontinuous right hand side) play an important role in many mechanical and engineering applications (e.g., see [1]), and present deep and complex mathematical questions. In particular, the well established Filippov convexification method (see [6]) gives a powerful mean to establish what to do when solution trajectories reach a co-dimension 1 manifold of discontinuity, but it is still not fully understood what happens when trajectories have to move on the intersection Σ of two smooth manifolds. To be of practical interest, such intersection Σ should enjoy some *attractivity* properties, that is nearby solution trajectories should reach Σ (in forward time), and solution trajectories starting on Σ ought to remain there, giving rise to so-called sliding motion. In [4], we characterized attractivity of Σ in the case of solution trajectories approaching it through sliding (see below). Our goal in this work is to complete the characterization of attractive Σ by treating the case of spiral attractivity of Σ. In short, our goal in this work is to give conditions characterizing a situation such as in Fig. 1, where Σ is the vertical axis; the top (red) portion is motion out of Σ, the bottom (blue) is motion toward Σ, and there is sliding motion on (part of) Σ (green curve), which itself is spirally-attractive.

In this Introduction, we review the basic problem and set up the notation. In Section 2 we propose a characterization of what we mean by spiral-like behavior around Σ, where Σ is the intersection of two smooth co-dimension 1 manifolds. Then, in Section 3 we characterize spiral attractivity for Σ. In Section 4 we discuss what may happen when, still subject to spiral-like behavior of nearby dynamics, Σ loses attractivity.

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1.1. The problem and Filippov solutions

We consider piecewise smooth differential systems of the following type:

$$\dot{x} = f(x), \quad f(x) = f_i(x), \quad x \in R_i, \quad i = 1, \dots, 4, \tag{1.1}$$

with initial condition $x(0) = x_0 \in R_i$, for some i. Here, the $R_i \subseteq \mathbb{R}^n$ are open, disjoint and connected sets, and (locally) $\mathbb{R}^n = \overline{\bigcup_i R_i}$. Each f_i is assumed to be smooth in an open neighborhood of the closure of each $R_i, i = 1, \dots, 4$.

Clearly, from (1.1), the vector field is not defined on the boundaries of the R_i's.

1.2. Codimension 1 case: attractivity, existence and uniqueness

The classical Filippov theory (see [6]) is concerned with the case of two regions R_1 and R_2, separated by a manifold Σ defined as the 0-set of a smooth ($\mathcal{C}^2$) scalar valued function h:

$$\begin{aligned} &\dot{x} = f_1(x), \quad x \in R_1, \quad \text{and} \quad \dot{x} = f_2(x), \quad x \in R_2, \\ \Sigma \quad &:= \{x \in \mathbb{R}^n : h(x) = 0\}, \quad h : \mathbb{R}^n \to \mathbb{R}. \end{aligned} \tag{1.2}$$

Here, ∇h is bounded away from 0 for all $x \in \Sigma$, and near Σ. Without loss of generality, we can label R_1 such that $h(x) < 0$ for $x \in R_1$, and R_2 such that $h(x) > 0$ for $x \in R_2$. Let us define

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} := \begin{bmatrix} \nabla h(x)^T f_1(x) \\ \nabla h(x)^T f_2(x) \end{bmatrix}, \quad x \in \Sigma, \tag{1.3}$$

for the projected vector fields. We say (see [6]) that Σ is *attractive* (in finite time) if, for some positive constant c, we have

$$w_1(x) \geqslant c > 0 \quad \text{and} \quad w_2(x) \leqslant -c < 0,$$

for $x \in \Sigma$. In this case, trajectories starting near Σ must reach it and remain there: this gives the so-called *sliding motion*. Filippov convexification method amounts to selecting as sliding vector field on Σ a convex combination of f_1 and f_2, $f_F := (1-\alpha)f_1 + \alpha f_2$, with α chosen so that $f_F \in T_\Sigma$ (f_F is tangent to Σ at each $x \in \Sigma$):

$$x' = (1-\alpha)f_1 + \alpha f_2, \quad \alpha = \frac{\nabla h(x)^T f_1(x)}{\nabla h(x)^T f_1(x) - \nabla h(x)^T f_2(x)}. \tag{1.4}$$

Clearly, because of attractivity, $\alpha \in (0,1)$. Whenever $\alpha = 0$ (respectively $\alpha = 1$), the vector field f_1 (respectively f_2), is itself tangent to Σ, and one should expect the trajectory to leave Σ to enter in R_1 (respectively R_2). These are tangential exits, predicted by the first order Filippov theory.

Remark 1.1. Observe that (1.4) gives a well defined sliding motion also in the case of *repulsive* Σ, that is when

$$w_1(x) \leqslant -c < 0 \quad \text{and} \quad w_2(x) \geqslant c > 0, \quad x \in \Sigma.$$

The difference in this case of repulsive sliding is that, for forward time, the sliding motion is unstable and there is no uniqueness, since one can also leave Σ at any instant of time with either f_1 or f_2. These types of exits are non-tangential.

1.3. Intersection of two codimension 1 manifolds

As we said, we are concerned with (1.1), where now the R_i's are (locally) separated by two intersecting smooth manifolds of co-dimension 1, $\Sigma_1 = \{x : \ h_1(x) = 0\}$ and $\Sigma_2 = \{x : \ h_2(x) = 0\}$. We have $\Sigma = \Sigma_1 \cap \Sigma_2$, and here h_1 and h_2 are $\mathcal{C}^2$ functions, and $\nabla h_1(x)$ and $\nabla h_2(x)$ are linearly independent, for x on (and in a neighborhood of) Σ.

We have four different regions R_1, R_2, R_3 and R_4 with the four different vector fields $f_i, i = 1, \dots, 4$, in these regions:

$$\dot{x} = f_i(x), \quad x \in R_i, \quad i = 1, \dots, 4. \tag{1.5}$$

Without loss of generality, we can label the regions as follows:

$$\begin{aligned} &R_1 : f_1 \quad \text{when} \quad h_1 < 0, h_2 < 0, \quad R_2 : f_2 \quad \text{when} \quad h_1 < 0, h_2 > 0, \\ &R_3 : f_3 \quad \text{when} \quad h_1 > 0, h_2 < 0, \quad R_4 : f_4 \quad \text{when} \quad h_1 > 0, h_2 > 0. \end{aligned} \tag{1.6}$$

We further let (cfr. with (1.3))

$$\begin{aligned} w_1^1 &= \nabla h_1^T f_1, \quad w_2^1 = \nabla h_1^T f_2, \quad w_3^1 = \nabla h_1^T f_3, \quad w_4^1 = \nabla h_1^T f_4, \\ w_1^2 &= \nabla h_2^T f_1, \quad w_2^2 = \nabla h_2^T f_2, \quad w_3^2 = \nabla h_2^T f_3, \quad w_4^2 = \nabla h_2^T f_4. \end{aligned} \tag{1.7}$$

In [4], the authors considered the case of Σ being *attractive through sliding*. By that, it is meant that solution trajectories starting near Σ will reach (in finite time) the intersection Σ, either directly, or (more likely) by first sliding on one of Σ_1 or Σ_2,