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Uniform stress field inside an anisotropic non-elliptical inhomogeneity interacting with a screw dislocation

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A B S T R A C T

We find that an anisotropic non-elliptical inhomogeneity interacting with a screw dislocation in a matrix subjected to remote uniform stresses and anti-plane shear deformations may still admit an internal uniform stress field. Our analysis indicates that the screw dislocation does not affect the uniform stresses inside the inhomogeneity but does play a crucial role in determining the shape of the inhomogeneity. In fact, we find that the influence of the screw dislocation gives rise to the possibility that the boundary of the inhomogeneity may have up to three sharp corners. Our discussion extends to the case when multiple screw dislocations interact with the non-elliptical inhomogeneity leading to a conjecture concerning the maximum allowable number of sharp corners in an inhomogeneity with Eshelby's uniformity property.

1. Introduction

Eshelby's seminal work (Eshelby, 1957, 1959, 1961) in the area of elastic inclusions and inhomogeneities continues to stimulate challenging and exciting research in this important area of composite mechanics. A comprehensive survey of recent works in this area can be found in Zhou et al. (2013). In particular, Eshelby's uniformity property which examines conditions under which elastic inclusions or inhomogeneities may maintain uniform interior stress distributions has been the subject of intense investigations in the recent literature (see, for example, Sendeckyj, 1970; Ru and Schiavone, 1996; Lubarda and Markenscoff, 1998; Liu, 2008; Kang et al., 2008; Wang, 2012; Wang and Schiavone, 2016; Dai et al., 2015, 2016). The practical importance of the uniformity property lies in the fact that a uniform interior stress distribution eliminates the possibility of stress peaks which are well-known to be responsible for the failure of the inhomogeneity. Accordingly, Eshelby's uniformity property is an optimal design criterion. Remarkably, it was shown recently that it is still possible to achieve uniform stresses inside a single or even pair of isotropic elastic inhomogeneities despite the influence of a screw dislocation introduced in the vicinity of the inhomogeneities (Wang and Schiavone, 2016; Dai et al., 2016).

In this paper, we are particularly interested in whether Eshelby's uniformity property remains valid when a screw dislocation interacts with an anisotropic elastic non-elliptical inhomogeneity inserted into an infinite matrix subjected to remote uniform anti-plane shear stresses. We begin by introducing a novel mapping function which includes a logarithmic term to account for the existence of the screw dislocation. The analysis indicates that the shape of the inhomogeneity required to induce an internal uniform stress field depends on the nearby screw dislocation while the internal uniform stress field itself does not. Detailed numerical results are presented to demonstrate our findings. Further, we extend our method to the study of the uniformity of stresses inside an anisotropic non-elliptical inhomogeneity interacting with multiple screw dislocations. Our results lead us to conjecture that when an anisotropic non-elliptical inhomogeneity interacts with M screw dislocations, the inhomogeneity will maintain its uniformity property with at most $M+2$ sharp corners appearing on its boundary. We confirm the validity of this conjecture in the cases $M = 1, 2, 3, 4$. Interestingly, the stresses in the matrix remain bounded at each of these sharp corners.

2. Formulation

For anti-plane shear deformations of a linearly anisotropic material possessing a plane of symmetry at $x_3 = 0$, the stress components σ_{31}, σ_{32}, the anti-plane displacement u_3 and the stress function ϕ can be expressed in terms of a single analytic function $f(z_p)$ as (Ting, 1996)

$$\sigma_{31} + p\sigma_{32} = \mathrm{i}\mu\,\mathrm{Im}\{p\}\overline{f'(z_p)}, \tag{1}$$

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$$\phi + i\mu u_3 = \mu f(z_p), \tag{2}$$

where $\mu = \sqrt{C_{44}C_{55} - C_{45}^2}$ with C_{44}, C_{55}, C_{45} being the elastic stiffnesses, $z_p = x_1 + px_2$ and

$$p = \frac{-C_{45} + i\sqrt{C_{44}C_{55} - C_{45}^2}}{C_{44}}. \tag{3}$$

The stresses σ_{31} and σ_{32} are related to the stress function ϕ through

$$\sigma_{31} = -\phi_{,2}, \quad \sigma_{32} = \phi_{,1}. \tag{4}$$

Let t_3 be the component of the anti-plane surface traction on a boundary L. If s is the arc-length measured along L such that the material remains on the left-hand side with the direction of increasing s, it can be shown that (Ting, 1996)

$$t_3 = -\frac{d\phi}{ds}. \tag{5}$$

Consider a domain in $\mathfrak{R}^2$, infinite in extent, containing a non-elliptical elastic inhomogeneity whose elastic properties differ from those of the surrounding matrix. The linearly anisotropic elastic materials occupying both the inhomogeneity and the matrix are assumed to be homogeneous and monoclinic with the plane of symmetry at $x_3 = 0$ and with the associated non-trivial elastic constants $C_{44}^{(1)}, C_{45}^{(1)}, C_{55}^{(1)}$ and $C_{44}^{(2)}, C_{45}^{(2)}, C_{55}^{(2)}$, respectively. The matrix is subjected to remote uniform anti-plane shear stresses $(\sigma_{31}^{\infty}, \sigma_{32}^{\infty})$ and a screw dislocation with Burgers vector b_3 applied at $(x_1, x_2) = (x_1^0, x_2^0)$. We represent the region occupied by the matrix by the domain S_2 and assume that the inhomogeneity occupies the region S_1. The inhomogeneity and the matrix are perfectly bonded together through the non-elliptical interface L. In what follows, the subscripts 1 and 2 will be used to identify the respective quantities in S_1 and S_2. To avoid confusion, we adopt the notation that $z = x_1 + ix_2$ and $z_j = x_1 + p_j x_2,\ j = 1,2$.

3. General solution

The corresponding boundary value problem takes the following form

$$\begin{aligned}
&f_2(z_2) + \overline{f_2(z_2)} = \Gamma f_1(z_1) + \Gamma\overline{f_1(z_1)},\\
&f_2(z_2) - \overline{f_2(z_2)} = f_1(z_1) - \overline{f_1(z_1)}, \quad z \in L;\\
&f_2(z_2) \cong \tfrac{b_3}{2\pi}\ln(z_2 - z_0^{(2)}) + O(1), \quad z_2 \to z_0^{(2)};\\
&f_2(z_2) \cong \tfrac{i\sigma_{31}^{\infty} + i\bar{p}_2\sigma_{32}^{\infty}}{\mu_2 \operatorname{Im}\{p_2\}} z_2 + \tfrac{b_3}{2\pi}\ln z_2 + O(1), \quad |z_2| \to \infty,
\end{aligned} \tag{6}$$

where $\Gamma = \mu_1/\mu_2$ and $z_0^{(2)} = x_1^0 + p_2 x_2^0$. The first two conditions in Eq. (6) represent the continuity of traction and displacement across the inhomogeneity-matrix interface L whereas the third condition characterizes the (singular) logarithmic behavior of $f_2(z_2)$ due to the presence of the screw dislocation at $z_2 = z_0^{(2)}$. Finally, the fourth condition in Eq. (6) describes the remote asymptotic behavior of $f_2(z_2)$.

It is convenient to introduce the following mapping function for the matrix

$$z_2 = \omega(\xi) = R\left(\xi + \frac{m}{\xi} + q\ln\frac{\xi - \bar{\xi}_0^{-1}}{\xi}\right), \quad \xi = \omega^{-1}(z_2), \quad |\xi| \ge 1, \tag{7}$$

where R is a real scaling constant, m and q are complex constants, and $\xi_0 = \omega^{-1}(z_0^{(2)})$. The branch cut for the logarithmic function $\ln\frac{\xi - \bar{\xi}_0^{-1}}{\xi}$, which is included in the mapping function to account for the screw dislocation, is chosen as the line segment connecting $\xi = 0$ and $\xi = \bar{\xi}_0^{-1}$. The presence of the logarithmic term in the mapping function clearly indicates that the inhomogeneity is non-elliptical. We mention that the mapping function in Eq. (7) maps the inhomogeneity-matrix interface L onto the unit circle $|\xi| = 1$; the region outside L (i.e. the matrix) onto the region outside the unit circle in the ξ-plane; and the screw dislocation located at $z_2 = z_0^{(2)}$ to the point $\xi = \xi_0$ in the ξ-plane. Furthermore, we remark that if we expand $\ln\frac{\xi - \bar{\xi}_0^{-1}}{\xi}$, $|\xi| \ge 1$ into convergent Laurent series with negative powers of ξ, the mapping function in Eq. (7) becomes the standard mapping function in Eq. (3.12-3) of Ting (1996).

In order to guarantee that the stress field inside the inhomogeneity is uniform, the analytic function defined inside the inhomogeneity takes the following form:

$$f_1(z_1) = \frac{A}{R} z_1, \quad z \in S_1, \tag{8}$$

where A is a complex number to be determined.

The enforcement of boundary conditions describing continuity of traction and displacement across the inhomogeneity-matrix interface L leads to

$$\begin{aligned}
f_2(\xi) = f_2(\omega(\xi)) &= \tfrac{1}{4}[A(\Gamma + 1)(1 - i\hat{p}) + \bar{A}(\Gamma - 1)(1 - i\bar{\hat{p}})]\left(\xi + \tfrac{m}{\xi} + q\ln\tfrac{\xi - \bar{\xi}_0^{-1}}{\xi}\right)\\
&+ \tfrac{1}{4}[A(\Gamma + 1)(1 + i\hat{p}) + \bar{A}(\Gamma - 1)(1 + i\bar{\hat{p}})]\left(\tfrac{1}{\xi} + \bar{m}\xi + \bar{q}\ln(\xi - \xi_0)\right), \quad |\xi| \ge 1,
\end{aligned} \tag{9}$$

where

$$\hat{p} = \frac{p_1 - p'_2}{p''_2}, \tag{10}$$

with p'_2 and p''_2, respectively, the real and complex parts of p_2.

The asymptotic behavior of $f_2(\xi)$ at infinity requires the following relationship

$$\begin{aligned}
&\frac{A}{R}(\Gamma + 1)[(1 - i\hat{p}) + \bar{m}(1 + i\hat{p})] + \frac{\bar{A}}{R}(\Gamma - 1)[(1 - i\bar{\hat{p}}) + \bar{m}(1 + i\bar{\hat{p}})]\\
&= \frac{4i(\sigma_{31}^{\infty} + \bar{p}_2\sigma_{32}^{\infty})}{\mu_2 \operatorname{Im}\{p_2\}},
\end{aligned} \tag{11}$$

which implies that the uniform stress field inside the anisotropic inhomogeneity is independent of the screw dislocation.

The complex constant A can be uniquely determined from Eq. (11) as

$$\frac{A}{R} = \frac{\begin{aligned}&4\{(\Gamma + 1)[(i - \bar{\hat{p}}) + m(i + \bar{\hat{p}})](\sigma_{31}^{\infty} + \bar{p}_2\sigma_{32}^{\infty})\\ &+ (\Gamma - 1)[(i + \bar{\hat{p}}) + \bar{m}(i - \bar{\hat{p}})](\sigma_{31}^{\infty} + p_2\sigma_{32}^{\infty})\}\end{aligned}}{\mu_2 \operatorname{Im}\{p_2\}\left(\begin{aligned}&(\Gamma + 1)^2[|1 + m|^2 + 2(1 - |m|^2)\operatorname{Im}\{\hat{p}\} + 4\operatorname{Im}\{m\}\operatorname{Re}\{\hat{p}\}\\ &+ |\hat{p}|^2|1 - m|^2]\\ &- (\Gamma - 1)^2[|1 + m|^2 - 2(1 - |m|^2)\operatorname{Im}\{\hat{p}\} + 4\operatorname{Im}\{m\}\operatorname{Re}\{\hat{p}\}\\ &+ |\hat{p}|^2|1 - m|^2]\end{aligned}\right)}. \tag{12}$$

The singular behavior of $f_2(\xi)$ at $\xi = \xi_0$ results in

$$q = \frac{2b_3}{\pi[A(\Gamma - 1)(1 - i\hat{p}) + \bar{A}(\Gamma + 1)(1 - i\bar{\hat{p}})]}, \tag{13}$$

which indicates that the complex number q depends on the Burgers vector of the screw dislocation and the prescribed remote stresses.

Furthermore, in order to ensure that the mapping function in Eq. (7) is one-to-one for $|\xi| > 1$, we must have $\omega'(\xi) \ne 0$ for $|\xi| > 1$. Equivalently, all three roots of the following cubic equation in ξ should lie within or on the unit circle

$$\xi^3 - \bar{\xi}_0^{-1}\xi^2 - (m - q\bar{\xi}_0^{-1})\xi + m\bar{\xi}_0^{-1} = 0. \tag{14}$$

Our specific results indicate that the condition that $\omega'(\xi) \ne 0$ for $|\xi| > 1$ is only necessary but not sufficient in order to ensure a one-to-