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The effect of surface bending and surface stress on the transmission of a vertical line force in soft materials

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ABSTRACT

When the surface of a soft substrate that carries a constant in-plane residual stress is indented by a concentrated line force, its profile near the applied load is found to have a kink, which results from a local balance of the surface stresses and the imposed force. Although the local bulk stresses in the substrate no longer have a net contribution to this force balance, they nevertheless grow according to a weak logarithmic singularity with respect to distance from the line load. Here we study how a normal line load is transmitted across a solid surface that can provide additional resistance due to bending deformation; we present an exact closed-form solution. Our analysis shows that the ability of the surface to resist bending completely regularizes the stress field — it is continuously differentiable everywhere. In particular, the stress state in the elastic substrate is hydrostatic right underneath the line load if the material is incompressible. Its maximum value is directly proportional to the applied normal load and inversely proportional to the elasto-bending length. It also depends on a dimensionless parameter which is the ratio of the elasto-capillary length to the elasto-bending length.

1. Introduction

Several studies over the past decade [1–3] have established the important and often dominant role of the surface of soft solids in their mechanical response. By far, the most commonly studied case is one in which the surface carries a constant and isotropic in-plane residual stress. Models for many canonical phenomena involving forces applied to surfaces of soft solids, such as the wetting by a liquid drop [4,5], and indentation of soft elastic substrates [6–8] have been re-examined and found to be qualitatively altered. Much less attention has been paid to more complex surface properties such as surface elasticity (strain-dependent resistance to stretching) and surface bending (resistance to surface curvature), although these have both been proposed theoretically [9–13] and are clearly present in some systems [14,15]. A fundamental problem that forms the basis for analysis of more complex phenomena is the response of a surface to a line load. In this work we study how a surface that, in addition to an in-plane stress carries bending energy, affects the transmission of a line force acting on it.

In wetting, the vertical component of a droplet's surface tension (locally a line load) pulls the surface of the soft substrate upwards, leading to the formation of a ridge at and near the applied line load [5,16]. When the role of the surface is negligible, a classical result of the theory of elasticity predicts that the local displacement under a line load has a logarithmic singularity, which results in a local stress field with a $1/r$ singularity, where r is the distance from the line load [17]. That is, within this theory, the displacement under the applied load diverges logarithmically with distance from it. However, for single-phase soft materials that carry a constant surface stress and have no bending resistance, if the elasto-capillary length is larger than molecular dimensions, experiments and theory [5,16,18,19] show that the local ridge geometry is a kink with finite displacement. The local angles of the lines that meet at the kink are described by balance of surface stresses at the contact line –Neumann's triangle of forces – where the surface stresses of the solid balance the force exerted by the contact line. This local picture is shown in Fig. 1 for the special case where the entire surface has the same isotropic surface stress or surface tension, σ. Hence, locally, the line force N is usually transmitted fully to the surfaces, and theory predicts that the displacements are bounded and the stresses have a weak logarithmic singularity.

The line load problem has been studied theoretically by several research groups [1,19–22]. These studies assumed: (1) small deformation based on linearized theory of elasticity, (2) the surface can be treated as a membrane with *isotropic surface stress* $\sigma \boldsymbol{I}_s$, where $\boldsymbol{I}_s$ the isotropic surface tensor and σ is the magnitude of (tensile) surface stress (often called *surface tension*) which is assumed to be independent of the surface stretch. More relevant to this work, the

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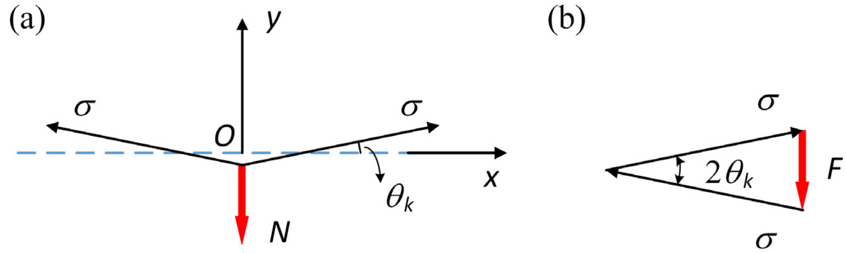

Fig. 1. Balance of forces at the kink formed on a soft solid when a line load is applied to it. The soft solid occupies the half space below the dashed blue line in (a). (a) Local geometry of kink. The deformed surface has a kink right underneath the line load. The vertical component of the surface tension, N, balances the applied force. (b) Local force balance requires that the surface stress σ and the applied line force N form a closed triangle, which is commonly known as the Neumann triangle.

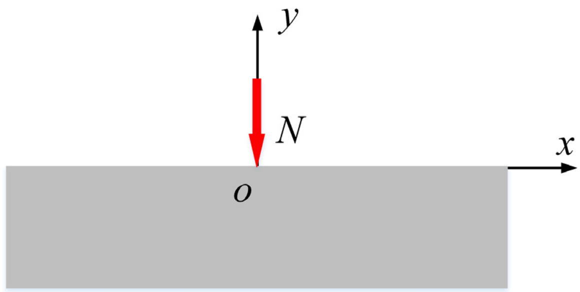

Fig. 2. Schematic of loading geometry. The elastic body occupies the lower half space y <0. A compressive line load of magnitude N >0 is imposed at the origin. N has units of force per unit length.

surface is assumed to have no bending rigidity. Here we note that most of the experimental studies of elasto-capillary phenomenon have focused on hydrogels where these assumptions are reasonable. Gurtin and Ian Murdoch [9] have raised the possibility that surfaces of elastic solids can store energy in bending but this issue has not received much attention. The exception is the work by Steigmann and Ogden [11] who develop constitutive models for surface with bending resistance. There are several examples of soft interfaces that can support both bending and tension. An example which is relevant to biology is the lipid bilayer where resistance to stretching is high, deformations of the bilayer generally conserve area, and the strain energy density of the interface is dominated by bending [13,23]. Kusumaatmaja et al. [15] have shown that the mechanics of the contact line between lipid bilayer membranes is governed by both surface bending and stress. A second example is where a new phase separates a soft solid from the air — e.g., a silica film a few nm thick that forms on the surface of an elastomer (e.g., polydimethylsiloxane) exposed to UV ozone or oxygen plasma [24,25]. These examples motivate us to study how an elastic substrate with surface bending and stress alters the transmission of force.

The plan of the paper is as follows. Section 2 states and formulates the problem. The exact solution is presented and results are discussed in Section 3. Section 4 concludes with a short summary.

2. Problem statement and formulation

The geometry is shown in Fig. 2 where an infinite block of a linear-elastic solid occupies the lower half space $|x| < \infty, y < 0$. The elasticity of the block is specified by its shear modulus μ and Poisson's ratio ν. Instead of a line load, we consider the more general situation where the surface at $y = 0$ is subjected to a *pressure load* $p_A(x)$, with no applied shear traction. This applied pressure is independent of the coordinate out of the plane of this page, i.e., perpendicular to x and y axis in Fig. 2 and we assume plane strain deformation where the out-of-plane displacement vanishes and the in-plane horizontal and vertical displacements u_1, u_2 are functions of the in-plane coordinates x and y only. We denote the in-plane stress and strain tensors in the elastic half space by $\sigma_{\alpha\beta}, \varepsilon_{\alpha\beta}$ respectively, where $\alpha, \beta = 1{,}2$.

In contrast to classical elasticity, the surface of the half space can resist deformation by bending and stretching. The bending stiffness of the surface is denoted by D. Stretching of the surface is resisted by surface tension σ, which we assume to be constant independent of surface stretch, as in previous works [7]. The change in curvature of the surface due to bending and stretching leads to a pressure *jump* of $-p_A(x) - \sigma_{22}\left(x, y = 0^-\right)$ across the interface. Here we use the standard convention that pressure is *negative* when tensile. This pressure jump is resisted by bending and stretching of the surface, and a simple force balance leads to

$$-p_A(x) - \sigma_{22}\left(x, y = 0^-\right) = D\frac{d^4 u_2(x, y_2 = 0)}{dx^4} - \sigma\frac{d^2 u_2(x, y = 0)}{dx^2}, \quad |x| < \infty \quad (1)$$

The first term on the RHS of (1) represents the pressure supported by bending, whereas the second term accounts for the curvature induced Laplace pressure due to surface tension. Thus, the behavior is governed by three materials parameters: D, σ, and μ. Absent the surface properties, D and σ, the problem of a response to a line force has no length scale associated with it in the sense that displacements and strains are everywhere proportional to the ratio N/μ. The introduction of the two surface properties, both of which are based on changes in surface shape, introduces two corresponding length scales that define distances over which the nature of the solution is altered. The characteristic distance over which bending alters the standard elasticity solution is given by $l_b = (\beta D/2\mu)^{1/3}$, where we call l_b the elasto-bending length. Here, $\beta \equiv 2(1-\nu)$, is introduced into this definition for later convenience. (For the common special case of an incompressible soft solid, $\beta = 1$.) We also define an elasto-capillary length $l_c = \beta\sigma/2\mu$, which defines a characteristic distance over which surface stress alters the elasticity solution. Thus, far from the line load, one expects standard elasticity always dominates (for a sufficiently large body.) Potentially, there are two other regions, one close to the line load where bending dominates and a second transition region at distances larger than l_b but less than l_c where surface stress dominates. Of course, whether the intermediate region exists is governed by the dimensionless ratio $\kappa \equiv l_c/l_b$, which needs to be significantly larger than unity for the existence of a region where surface stress dominates. It is evident that previous theories [7], which ignore bending, can be obtained by setting $D = 0$ in (1). Our definition of κ is similar in spirit to the parameter used by Charlotte et al. [26] and Paulsen et al. [27] (in our notation their parameter is $\sqrt{D/\sigma}$) who consider capillary wrapping of thin elastic sheets. Specifically, when κ is small, bending dominates and the effect of surface stress is not important. On the other hand, for large κ, bending dominates in a very small region underneath the line load, outside this region, surface stress dominates.